

Form factors of integrable higher-spin XXZ chains and the affine quantum-group symmetry

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Abstract

We derive exactly scalar products and form factors for integrable higher-spin XXZ chains through the algebraic Bethe-ansatz method. Here spin values are arbitrary and different spins can be mixed. We show the affine quantum-group symmetry, $U_q(\widehat{sl_2})$, for the monodromy matrix of the XXZ spin chain, and then obtain the exact expressions. Furthermore, through the quantum-group symmetry we explicitly derive the diagonalized forms of the B and C operators in the F -basis for the spin-1/2 XXZ spin chain, which was conjectured in the algebraic Bethe-ansatz calculation of the XXZ correlation functions. The results should be fundamental in studying form factors and correlation functions systematically for various solvable models associated with the integrable XXZ spin chains.

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1 Introduction

Correlation functions of the spin-1/2 XXZ spin chain have attracted much attention in mathematical physics for more than a decade [1, 2, 3]. The multiple-integral representations of XXZ correlation functions were first derived in terms of the q -vertex operators [4]. Based on the algebraic Bethe ansatz method, the determinant expressions [5] of the scalar products and the norms of Bethe ansatz eigenstates were reconstructed in terms of the F -basis [6], and then the XXZ correlation functions are derived under an external magnetic field [7, 8]. The multiple-integral representations at zero temperature have been extended into those at finite temperature [9]. Furthermore, dynamical structure factors have been evaluated by solving the Bethe ansatz equations numerically [10, 11, 12, 13].

The Hamiltonian of the spin-1/2 XXZ chain under the periodic boundary conditions is given by

$$\mathcal{H}_{\text{XXZ}} = \frac{1}{2} \sum_{j=1}^L (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z). \quad (1.1)$$

Here σ_j^a ($a = X, Y, Z$) are the Pauli matrices defined on the j th site, and Δ the XXZ coupling. By $\Delta = (q + q^{-1})/2$ we define parameter q , which plays a significant role in the paper. We note that L denotes the number of the one-dimensional lattice sites. Solvable higher-spin generalizations of the XXX and XXZ chains have been constructed by the fusion method in several references [14, 15, 16, 17, 18, 19]. For instance, the Hamiltonian of the solvable spin- s XXX chain is given by the following [15]:

$$H_s = J \sum_{j=1}^L Q_{2s}(\vec{S}_j \cdot \vec{S}_{j+1}), \quad (1.2)$$

where \vec{S}_j are operators of spin s acting on the j th site and $Q_{2s}(x)$ is a polynomial of degree $2s$

$$Q_{2s}(x) = \sum_{p=1}^{2s} \left(\sum_{k=1}^p \frac{1}{k} \right) \prod_{\ell=0, \ell \neq j}^{2s} \frac{x - x_\ell}{x_j - x_\ell}. \quad (1.3)$$

Here $x_\ell = [\ell(\ell+1) - 2s(s+1)]/2$. At $T = 0$ in the critical regime, it is discussed that the low-excitation spectrum of the spin- s XXX chain is described in terms of the level- k $SU(2)$ WZWN model where $k = 2s$ [20].

In the present paper we derive exact expressions of scalar products and form factors for the integrable higher-spin XXZ spin chains. Here different spins can be mixed. We first show that the monodromy matrix of the XXZ spin chain has the symmetry of the affine quantum group $U_q(\widehat{sl_2})$. Then, we derive the exact expressions taking advantage of the quantum group symmetry. By a similarity transformation [21], we transform the symmetric R -matrix into an asymmetric one, which is directly connected to the quantum group $U_q(sl_2)$. We derive projection operators from the asymmetric R -matrices [19], and construct integrable higher-spin XXZ spin chains by the fusion method similarly as the case of the XXX spin chain [14]. Here we make an extensive use of the q -analogues of Young's projection operators, which play a central role in the q -analogue of the Schur-Weyl reciprocity of the quantum group $U_q(sl_2)$ [22, 23]. Hereafter, we call transformations

on the R -matrix gauge transformations. After we construct integrable higher-spin models, applying the inverse transformation to them, we reduce the asymmetric monodromy matrices into those of the symmetric L -operators constructed from the symmetric R -matrices, and thus obtain scalar products and form factors for the integrable higher-spin models in the standard formulation.

Form factors and correlation functions have been discussed for integrable higher-spin XXX models in previous researches [24, 25, 26, 27]. In the approach of the so-called quantum inverse scattering problem for the integrable N -state models, one has to construct the N -by- N monodromy matrix in order to express local operators in terms of the global operators. However, it seems to be technically nontrivial to construct the N -by- N monodromy matrix for the integrable higher-spin systems (see also [28]). By the approach of the present paper, the calculational task is much reduced into the minimal level. In fact, as a consequence of the q -analogue of the Schur-Weyl duality, the exact expressions of scalar products and form factors are derived from the formulas of the spin-1/2 XXZ chain by setting the inhomogeneous parameters in the form of “complete ℓ -strings”[29], and the most general results are straightforwardly obtained.

The affine quantum-group invariance has another important consequence. In the XXZ case we can explicitly prove the pseudo-diagonalized forms of the B and C operators of the algebraic Bethe-ansatz method through the quantum-group symmetry. Here we remark that the XXZ spin chain has no spin SU(2) symmetry. In the pseudo-diagonal basis, the B and C operators are expressed as sums of local spin operators σ_i^- and σ_i^+ multiplied by diagonal matrices, respectively, where each of the local spin operators σ_i^\pm are defined on one lattice-site, i.e. on the i th site. The pseudo-diagonalized forms of the B and C operators were conjectured in the algebraic Bethe-ansatz derivation of the XXZ correlation functions [6, 7, 8]. In fact, the B operators create the Bethe states where the C operators are conjugate to them, and the pseudo-diagonalized forms play a central role in the calculation of the scalar products and the norms of Bethe states. They are also fundamental in the quantum inverse scattering problem, by which local operators are expressed in terms of the global operators such as the B and C operators. In the XXZ case, however, an explicit derivation of the pseudo-diagonalized forms of the B and C operators has not been shown previously, yet. Thus, the explicit derivation in the paper completes the algebraic Bethe-ansatz formulation of the form factors and correlation functions of the integrable XXZ spin chains. Here we remark that for the XXX case (i.e. the isotropic case), the diagonalized forms have been shown by Maillet and Sanchez de Santos in Ref. [6] by making an explicit use of the rotational SU(2)-symmetry. However, the method for the XXX case does not hold for the XXZ spin chain which has no SU(2) symmetry.

The derivation of the affine quantum-group symmetry of the monodromy matrix should be not only theoretically interesting but also practically useful for calculation. Here we remark that the infinite-dimensional symmetry, $U_q(\widehat{sl}_2)$, was realized for the infinite XXZ spin chain with the q -vertex operators [2, 4]. Thus, it should be an interesting open problem how the affine quantum-group symmetry of the finite XXZ spin chain can be related to that of the infinite XXZ spin chain. Furthermore, there are several advantages in

the present formulation of the affine quantum-group symmetry. We derive the symmetry through gauge transformations. The transformed asymmetric R -matrix is directly related to the quantum group $U_q(sl_2)$ so that we can systematically construct higher-spin representations of the R -matrices through the q -analogue of the Young symmetrizers. Here we should note that the gauge transformation connects the R matrix in the different gradings of $U_q(\widehat{sl}_2)$. The symmetric and asymmetric R -matrices are equivalent to that of the principal and homogeneous gradings, respectively (see for instance §5.4 of [2]). Moreover, we thus avoid technical difficulties appearing when we directly derive the matrix representation of the universal R -matrix of the affine quantum group, which is given by a product of infinite series of generators [30]. Although one can construct matrix representations of the modified universal R -matrix with its derivation d dropped [30, 22](see A.2 of [31]), it seems that the calculation is not quite straightforward when we construct higher-spin representations.

The results of the present paper should be useful for calculating exact expressions of correlation functions for various integrable models associated with higher-spin XXZ chains. For instance, the τ_2 model in the N -dimensional nilpotent representation corresponds to the integrable spin- $(N-1)/2$ XXZ spin chain with q being a primitive N th root of unity [32]. Here we remark that the τ_2 model is closely related to the N -state superintegrable chiral Potts model. Furthermore, there are several possible physical applications, such as calculating form factors of quantum impurity models through the result in the case of mixed spins. As an illustrative example, we have calculated exact expressions for the emptiness formation probability of the higher-spin XXZ spin chains, which we shall discuss in a subsequent paper.

The content of the paper consists of the following: In section 2, we introduce the symmetric R -matrix of the spin-1/2 XXZ spin chain, and define the monodromy matrix [33]. We also define the action of the symmetric group on products of R -matrices. In section 3 we derive the symmetry of the quantum affine algebra for the monodromy matrix of the XXZ spin chain. We introduce the asymmetric R -matrix and then derive it from the symmetric one by a gauge transformation. We decompose the asymmetric R -matrix in terms of the generators of the Temperley-Lieb algebra, and show the affine quantum-group symmetry. We also derive it by a systematic method for expressing products of R -matrices formulated in definition 3. In fact, all the fundamental relations of the quantum inverse-scattering problem can be derived much more simply without using the \hat{R} -matrix of Ref. [6], as shown in Appendices A and B. In section 4 we construct the R matrices of integrable higher-spin XXZ spin chains with projection operators of $U_q(sl_2)$ by the fusion method. We also discuss the case of mixed spins. In section 5 we formulate an explicit derivation of the pseudo-diagonalized forms of the B -operators. We also show it for the C -operator in Appendix E. In section 6 we derive determinant expressions of scalar products for the higher-spin XXZ spin chains. In section 7, for the higher-spin cases we show the method by which we can express local operators in terms the global operators. We give some useful formulas of the quantum inverse scattering problem for the higher spin case. Finally, we derive some examples of form factors for the integrable higher-spin XXZ spin chains.

2 R -matrices and L -operators

2.1 Symmetric R -matrix

We shall introduce the R -matrix for the XXZ spin chain [33]. We consider two types of R -matrices, $R_{ab}(u)$ and $R_{ab}(\lambda, \mu)$. The R -matrix with a single rapidity argument, $R_{ab}(u)$, acts on the tensor product of two vector spaces V_a and V_b , i.e. $R_{ab}(u) \in \text{End}(V_a \otimes V_b)$, where parameter u is independent of V_a or V_b . The R -matrix with two rapidity arguments, $R_{ab}(\lambda, \mu)$, acts on the tensor product of vector spaces with parameters, $V_a(\lambda)$ and $V_b(\mu)$, i.e. $R_{ab}(\lambda, \mu) \in \text{End}(V_a(\lambda) \otimes V_b(\mu))$.

Let us denote by $e^{a,b}$ such a matrix that has only one nonzero element equal to 1 at entry (a, b) . We denote by V the two-dimensional vector space. We define the R -matrix acting on the tensor product $V \otimes V$ by

$$R(u) = \sum_{a,b,c,d=1,2} R_{cd}^{ab}(u) e^{a,c} \otimes e^{b,d}. \quad (2.1)$$

Here matrix elements $R_{cd}^{ab}(u)$ satisfy the charge conservation, i.e. $R_{cd}^{ab}(u) = 0$ unless $a + b = c + d$, and all the nonzero elements are given by the following:

$$\begin{aligned} R_{11}^{11}(u) &= R_{22}^{22}(u) = 1, & R_{12}^{12}(u) = R_{21}^{21}(u) &= b(u), \\ R_{21}^{12}(u) &= R_{12}^{21}(u) = c(u), \end{aligned} \quad (2.2)$$

where functions $b(u)$ and $c(u)$ are given by

$$b(u) = \frac{\sinh(u)}{\sinh(u + \eta)}, \quad c(u) = \frac{\sinh(\eta)}{\sinh(u + \eta)}. \quad (2.3)$$

Here, parameter η is related to q of $\Delta = (q + q^{-1})/2$ by $q = \exp(\eta)$.

We now introduce operators acting on the L th power of tensor product of vector spaces with parameters, $V(\lambda_1) \otimes \dots \otimes V(\lambda_L)$. We generalize the notation of (2.1). Let us take a pair of integers j and k satisfying $1 \leq j < k \leq L$. For a given set of matrix elements $A_{c,d}^{a,b}(\lambda_j, \lambda_k)$ ($a, b, c, d = 1, 2$) we define operators $A_{j,k}(\lambda_j, \lambda_k)$ and $A_{k,j}(\lambda_k, \lambda_j)$ by

$$\begin{aligned} A_{j,k}(\lambda_j, \lambda_k) &= \sum_{a,b,\alpha,\beta=1,2} A_{b,\beta}^{a,\alpha}(\lambda_j, \lambda_k) I_1 \otimes \dots \otimes I_{j-1} \\ &\quad \otimes e_j^{a,b} \otimes I_{j+1} \otimes \dots \otimes I_{k-1} \otimes e_k^{\alpha,\beta} \otimes I_{k+1} \otimes \dots \otimes I_L, \\ A_{k,j}(\lambda_k, \lambda_j) &= \sum_{a,b,\alpha,\beta=1,2} A_{\beta,b}^{\alpha,a}(\lambda_k, \lambda_j) I_1 \otimes \dots \otimes I_{j-1} \\ &\quad \otimes e_j^{a,b} \otimes I_{j+1} \otimes \dots \otimes I_{k-1} \otimes e_k^{\alpha,\beta} \otimes I_{k+1} \otimes \dots \otimes I_L. \end{aligned} \quad (2.4)$$

Here I is the two-by-two unit matrix, and I_j and $e_j^{a,b}$ act on the j th vector space $V(\lambda_j)$ of $V(\lambda_1) \otimes \dots \otimes V(\lambda_L)$. In terms of matrices we express operators $A_{j,k}$ and $A_{k,j}$ for $j < k$ by

$$A_{j,k} = \begin{pmatrix} A_{11}^{11} & A_{12}^{11} & A_{21}^{11} & A_{22}^{11} \\ A_{11}^{12} & A_{12}^{12} & A_{21}^{12} & A_{22}^{12} \\ A_{11}^{21} & A_{12}^{21} & A_{21}^{21} & A_{22}^{21} \\ A_{11}^{22} & A_{12}^{22} & A_{21}^{22} & A_{22}^{22} \end{pmatrix}_{[j,k]}, \quad A_{k,j} = \begin{pmatrix} A_{11}^{11} & A_{21}^{11} & A_{12}^{11} & A_{22}^{11} \\ A_{11}^{21} & A_{21}^{21} & A_{12}^{21} & A_{22}^{21} \\ A_{11}^{12} & A_{21}^{12} & A_{12}^{12} & A_{22}^{12} \\ A_{11}^{22} & A_{21}^{22} & A_{12}^{22} & A_{22}^{22} \end{pmatrix}_{[j,k]} \quad (2.5)$$

Here by the symbol $[j, k]$ we express that matrix element A_{cd}^{ab} corresponds to $e_j^{a,c} \otimes e_k^{b,d}$ for $A_{j,k}$, and to $e_j^{b,d} \otimes e_k^{a,c}$ for $A_{k,j}$.

Let us now introduce operators $R_{j,k}(\lambda_j, \lambda_k)$ and $R_{k,j}(\lambda_k, \lambda_j)$ acting on the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_L)$. We define them by putting $A_{cd}^{ab}(\lambda_j, \lambda_k) = R_{cd}^{ab}(\lambda_j - \lambda_k)$ in (2.4). Here the matrix elements $R_{cd}^{ab}(u)$ are given in (2.2). For instance, setting $u = \lambda_1 - \lambda_2$, we have explicitly

$$R_{12}(\lambda_1, \lambda_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]} . \quad (2.6)$$

The R -matrices satisfy the Yang-Baxter equations:

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2) \quad (2.7)$$

They also satisfy the inversion relations (unitarity conditions):

$$R_{jk}(\lambda_j, \lambda_k)R_{kj}(\lambda_k, \lambda_j) = I^{\otimes L} \quad \text{for } 1 \leq j, k \leq L . \quad (2.8)$$

Here $I^{\otimes L}$ denotes the L th power of tensor product of I .

Hereafter we often abbreviate $R_{jk}(\lambda_j, \lambda_k)$ simply by R_{jk} .

2.2 L -operators and the monodromy matrix

Let us introduce parameters $\xi_1, \xi_2, \dots, \xi_L$, which we call the inhomogeneous parameters. In the case of the monodromy matrix, we assume that parameters λ_j of the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_L)$ are given by the inhomogeneous parameters, i.e. $\lambda_j = \xi_j$ for $j = 1, 2, \dots, L$. Let us denote by 0 the suffix for the auxiliary space. We define L -operators acting on the m th site for $m = 1, 2, \dots, L$, by

$$L_m(\lambda, \xi_m) = R_{0m}(\lambda, \xi_m) . \quad (2.9)$$

We define the monodromy matrix acting on the L lattice-sites in one dimension by

$$T_{0,12\dots L}(\lambda; \xi_1, \dots, \xi_L) = L_L(\lambda, \xi_L)L_{L-1}(\lambda, \xi_{L-1}) \cdots L_2(\lambda, \xi_2)L_1(\lambda, \xi_1) . \quad (2.10)$$

We shall also denote it by $R_{0,12\dots L}(\lambda_0; \xi_1, \dots, \xi_L)$ in §2.4. Hereafter we often suppress the symbols of inhomogeneous parameters and express the monodromy matrix $T_{0,12\dots L}(\lambda; \xi_1, \dots, \xi_L)$ simply as $R_{0,12\dots L}(\lambda; \{\xi_j\})$ or $T_0(\lambda)$.

Let us consider two auxiliary spaces with suffices a and b . We define monodromy matrices $T_a(\lambda_a)$ and $T_b(\lambda_b)$ similarly as (2.10) with 0 replaced by a and b , respectively. It is clear that they satisfy the following Yang-Baxter equations.

$$R_{ab}(\lambda_a, \lambda_b)T_a(\lambda_a)T_b(\lambda_b) = T_b(\lambda_b)T_a(\lambda_a)R_{ab}(\lambda_a, \lambda_b) . \quad (2.11)$$

Let us introduce operator A_j acting on the j th site by

$$A_j = \sum_{a,b=1,2} A_j^a I_0 \otimes \cdots \otimes I_{j-1} \otimes e_j^{a,b} \otimes \cdots \otimes I_L \quad (2.12)$$

We express it in terms of the matrix notation as follows:

$$A_j = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix}_{[j]} \quad (2.13)$$

We express the matrix elements of the monodromy matrix by

$$T_{0,12\dots L}(u; \xi_1, \dots, \xi_L) = \begin{pmatrix} A_{12\dots L}(u; \xi_1, \dots, \xi_L) & B_{12\dots L}(u; \xi_1, \dots, \xi_L) \\ C_{12\dots L}(u; \xi_1, \dots, \xi_L) & D_{12\dots L}(u; \xi_1, \dots, \xi_L) \end{pmatrix}_{[0]} \quad (2.14)$$

The transfer matrix, $t(u)$, is given by the trace of the monodromy matrix with respect to the 0th space:

$$\begin{aligned} t(u; \xi_1, \dots, \xi_L) &= \text{tr}_0(T_{0,12\dots L}(u; \xi_1, \dots, \xi_L)) \\ &= A_{12\dots L}(u; \xi_1, \dots, \xi_L) + D_{12\dots L}(u; \xi_1, \dots, \xi_L). \end{aligned} \quad (2.15)$$

Here we note that the transfer matrix $t(u)$ is nothing but the transfer matrix of the six-vertex model defined on the two-dimensional square lattice [34].

Hereafter, we shall often denote $B_{12\dots L}(u; \xi_1, \dots, \xi_L)$ by $B(u; \{\xi_j\})$ or $B(u)$, briefly.

2.3 Products of R -matrices and the symmetric group

Let us consider the symmetric group \mathcal{S}_n of n integers, $1, 2, \dots, n$. We denote by σ an element of \mathcal{S}_n . Then σ maps j to $\sigma(j)$ for $j = 1, 2, \dots, n$

Definition 1. Let p be a sequence of n integers, $1, 2, \dots, n$, and σ an element of the symmetric group \mathcal{S}_n . We define the action of σ on p by

$$\sigma(p) = (p_{\sigma(1)}, \dots, p_{\sigma(n)}). \quad (2.16)$$

Here we remark that $(\sigma_A \sigma_B)p = \sigma_B(\sigma_A p)$ for $\sigma_A, \sigma_B \in \mathcal{S}_n$. We shall show it in Appendix A.

Let us recall that R_{jk} denote $R_{jk}(\lambda_j, \lambda_k)$.

Definition 2. Let $p = (p_1, p_2, \dots, p_n)$ be a sequence of n integers, $1, 2, \dots, n$. We define $R_{p_1, p_2 p_3 \dots p_n}$ and $R_{p_1 p_2 \dots p_{n-1}, p_n}$ by

$$\begin{aligned} R_{p_1, p_2 p_3 \dots p_n} &= R_{p_1 p_n} R_{p_1 p_{n-1}} \cdots R_{p_1 p_2}, \\ R_{p_1 p_2 \dots p_{n-1}, p_n} &= R_{p_1 p_n} R_{p_2 p_n} \cdots R_{p_{n-1} p_n}. \end{aligned} \quad (2.17)$$

For $p = (1, 2, \dots, n)$ we have

$$R_{1,23\dots n} = R_{1n} R_{1n-1} \cdots R_{12}, \quad R_{12\dots n-1,n} = R_{1n} R_{2n} \cdots R_{n-1n}. \quad (2.18)$$

We thus express the monodromy matrix as follows

$$T_{0,12\dots L}(\lambda_0; \{\xi_k\}) = R_{0,12\dots L}(\lambda_0; \{\xi_k\}). \quad (2.19)$$

Here we have assumed that $\lambda_k = \xi_k$ for $k = 1, 2, \dots, n$.

Let us express by $s_j = (j \ j+1)$ such a permutation that maps j to $j+1$ and $j+1$ to j and does not change other integers.

Definition 3. Let p be a sequence of n integers, $1, 2, \dots, n$. We define $R_p^{s_j}$ by

$$R_p^{s_j} = R_{p_j, p_{j+1}}(\lambda_{p_j}, \lambda_{p_{j+1}}). \quad (2.20)$$

For the unit element e of \mathcal{S}_n , we define R_p^e by $R_p^e = 1$. For a given element σ of \mathcal{S}_n , we define R_p^σ recursively by the following:

$$R_p^{\sigma_A \sigma_B} = R_{\sigma_A(p)}^{\sigma_B} R_p^{\sigma_A}. \quad (2.21)$$

We remark that every permutation σ is expressed as a product of some $s_j = (j \ j+1)$ with $j = 1, 2, \dots, n-1$. We thus obtain R_p^σ as a product of $R_p^{s_j}$ for some j 's. For an illustration, let us calculate $R_{(1,2,3)}^{(123)}$. Noting $(123) = (1\ 2)(2\ 3)$, we have

$$\begin{aligned} R_{(1,2,3)}^{(12)(23)} &= R_{(2,1,3)}^{(23)} R_{(1,2,3)}^{(12)} \\ &= R_{13} R_{12} = R_{1,23}. \end{aligned} \quad (2.22)$$

Through the defining relations of the symmetric group \mathcal{S}_n [35], we can show that definition 3 is well defined. The proof is given in proposition A.1 of Appendix A.

Let us denote by σ_c such a cyclic permutation that maps j to $j+1$ for $j = 1, \dots, n-1$ and n to 1. We also express it as $\sigma_c = (12 \cdots n)$. Noting $(12 \cdots n) = (1\ 2) \cdots (n-1\ n) = s_1 s_2 \cdots s_{n-1}$, we can show the following lemma

Lemma 4. Let us denote by p_q the sequence $p_q = (1, 2, \dots, n)$. For $\sigma_c = (12 \cdots n)$ we have

$$R_{p_q}^{\sigma_c} = R_{1,2 \cdots n}. \quad (2.23)$$

The proof of lemma 4 is given in lemma A.2 of Appendix A.

2.4 \check{R} -matrices and permutation operators

Let us consider two-dimensional vector spaces V_a and V_b . We define permutation operator Π_{ab} which maps elements of $V_a \otimes V_b$ to those of $V_b \otimes V_a$ as follows.

$$\Pi_{ab} v_a \otimes v_b = v_b \otimes v_a, \quad v_a \in V_a, \quad v_b \in V_b. \quad (2.24)$$

We define $\check{R}_{ab}(u)$ by

$$\check{R}_{ab}(u) = \Pi_{ab} R_{ab}(u) \quad (2.25)$$

The operators \check{R}_{ab} satisfy the Yang-Baxter equations

$$\check{R}_{12}(u) \check{R}_{23}(u+v) \check{R}_{12}(v) = \check{R}_{23}(v) \check{R}_{12}(u+v) \check{R}_{23}(u) \quad (2.26)$$

The operator \check{R}_{ab} gives a linear map from $V_a \otimes V_b$ to $V_b \otimes V_a$. If V_a and V_b are equivalent, then we may regard \check{R}_{ab} as a map from $V_a^{\otimes 2}$ to $V_a^{\otimes 2}$.

We add 0 to the n integers. For a given element σ of the symmetric group S_{n+1} , we define Π^σ acting on integers $0, 1, \dots, n$, as follows. We first express σ in terms of $s_j = (j \ j + 1)$ such as $\sigma = s_{j_1} s_{j_2} \cdots s_{j_r}$, and then we define $\Pi^\sigma = \Pi^{(j_1 j_1+1) \cdots (j_r j_r+1)}$ by

$$\Pi^\sigma = \Pi_{j_1, j_1+1} \Pi_{j_2, j_2+1} \cdots \Pi_{j_r, j_r+1}. \quad (2.27)$$

Lemma 5. *We have the following relation between R-matrices and operators \check{R}_{ab} :*

$$R_{0,12\cdots n} = \Pi^{(01\cdots n)} \check{R}_{n-1\,n}(\lambda_0 - \xi_{n-1}) \cdots \check{R}_{12}(\lambda_0 - \xi_2) \check{R}_{01}(\lambda_0 - \xi_1) \quad (2.28)$$

3 The quantum group invariance

We shall show that the monodromy matrix, $T_{0,12\cdots L}(\lambda; \xi_1, \dots, \xi_L)$, has the symmetry of the affine quantum group, $U_q(\widehat{sl}_2)$.

3.1 Quantum group $U_q(sl_2)$ and the asymmetric R-matrices

The quantum algebra $U_q(sl_2)$ is an associative algebra over \mathbf{C} generated by X^\pm, K^\pm with the following relations: [31]

$$\begin{aligned} KK^{-1} &= KK^{-1} = 1, \quad KX^\pm K^{-1} = q^{\pm 2} X^\pm, \quad , \\ [X^+, X^-] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned} \quad (3.1)$$

The algebra $U_q(sl_2)$ is also a Hopf algebra over \mathbf{C} with comultiplication

$$\begin{aligned} \Delta(X^+) &= X^+ \otimes 1 + K \otimes X^+, \quad \Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^-, \\ \Delta(K) &= K \otimes K, \end{aligned} \quad (3.2)$$

and antipode: $S(K) = K^{-1}, S(X^+) = -K^{-1}X^+, S(X^-) = -X^-K$, and coproduct: $\epsilon(X^\pm) = 0$ and $\epsilon(K) = 1$.

In association with the quantum group, we define the q -integer of an integer n by $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

The universal R-matrix, \mathcal{R} , of $U_q(sl_2)$ satisfies the following relations:

$$\mathcal{R}\Delta(x) = \tau \circ \Delta(x)\mathcal{R} \quad \text{for all } x \in U_q(sl_2). \quad (3.3)$$

Here τ denotes a permutation such that $\tau a \otimes b = b \otimes a$ for $a, b \in U_q(sl_2)$.

We now introduce some notation of a Hopf algebra. Let $x(1), \dots, x(n)$ be elements of Hopf algebra \mathcal{A} . For a given permutation σ of S_n , we define its action on the tensor product $x(1) \otimes \cdots \otimes x(n)$ as follows:

$$\sigma \circ (x(1) \otimes \cdots \otimes x(n)) = x(\sigma^{-1}1) \otimes \cdots \otimes x(\sigma^{-1}n). \quad (3.4)$$

We note that \mathcal{A} has coassociativity: $(\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x)$ for any element x of \mathcal{A} . We therefore denote it by $\Delta^{(2)}(x)$. We define $\Delta^{(n)}(x)$ recursively by

$$\Delta^{(n)}(x) = (\Delta^{(n-1)} \otimes id)\Delta(x) \quad \text{for } x \in \mathcal{A}. \quad (3.5)$$

Let us now introduce the following asymmetric R -matrices:

$$R^\pm(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^\mp(u) & 0 \\ 0 & c^\pm(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.6)$$

where $c^\pm(u)$ are defined by

$$c^\pm(u) = \frac{e^{\pm u} \sinh(\eta)}{\sinh(u + \eta)}. \quad (3.7)$$

In the spin-1/2 representation of $U_q(sl_2)$, we have the following relations:

$$R_{12}^+(u)\Delta(x) = \tau \circ \Delta(x)R_{12}^+(u) \quad \text{for } x = X^\pm, K. \quad (3.8)$$

Here we remark that spectral parameter u is arbitrary and independent of X^\pm or K . Similarly as in the symmetric case, we define the monodromy matrix $R_{0,1\dots n}^+$ by $R_{0,1\dots n}^+ = R_{0,n}^+ \cdots R_{0,1}^+$, and \check{R}^+ by $\check{R}_{12}^+(u) = \Pi_{12}R^+(u)$.

Lemma 6. *The monodromy matrix expressed in terms of \check{R} 's commutes with the action of the quantum group $U_q(sl_2)$:*

$$[\check{R}_{L-1,L}^+(\lambda - \xi_L) \cdots \check{R}_{1,2}^+(\lambda - \xi_2)\check{R}_{0,1}^+(\lambda - \xi_1), \Delta^{(L)}(x)] = 0, \quad \text{for all } x \in U_q(sl_2). \quad (3.9)$$

Here parameters $\lambda, \xi_1, \dots, \xi_L$ are independent of element x of $U_q(sl_2)$.

We shall show lemma 6 and eqs. (3.8) through the Temperley-Lieb algebra in §3.2.

Lemma 6 leads to the following symmetry relations of the monodromy matrix $R_{0,12\dots L}^+$ with respect to the quantum group $U_q(sl_2)$:

Proposition 7. *Let σ_c be a cyclic permutation: $\sigma_c = (01 \cdots L)$. Then we have*

$$R_{0,12\dots L}^+(\lambda; \xi_1, \dots, \xi_L) \Delta^{(L)}(x) = \sigma_c \circ \Delta^{(L)}(x) R_{0,12\dots L}^+(\lambda; \xi_1, \dots, \xi_L) \quad \text{for all } x \in U_q(sl_2) \quad (3.10)$$

Here parameters $\lambda, \xi_1, \dots, \xi_L$ are independent of element x of $U_q(sl_2)$.

Proof. Making use of lemma 6 we show (3.10) from (2.28) and the following relation:

$$\sigma_c \circ \Delta^{(L)}(x) = \Pi^{\sigma_c} \Delta^{(L)}(x) (\Pi^{\sigma_c})^{-1}. \quad (3.11)$$

□

3.2 Derivation in terms of the Temperley-Lieb algebra

Let us define U_j^\pm for $j = 0, 1, \dots, L - 1$, by

$$U_j^\pm = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & q^\mp & -1 & 0 \\ 0 & -1 & q^\pm & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)_{[j,j+1]}. \quad (3.12)$$

They satisfy the defining relations of the Temperley-Lieb algebra: [34]

$$\begin{aligned} U_j^\pm U_{j+1}^\pm U_j^\pm &= U_j^\pm, \\ U_{j+1}^\pm U_j^\pm U_{j+1}^\pm &= U_j^\pm, \quad \text{for } j = 0, 1, \dots, L-2, \\ \left(U_j^\pm \right)^2 &= (q + q^{-1}) U_j^\pm \quad \text{for } j = 0, 1, \dots, L-1, \\ U_j^\pm U_k^\pm &= U_k^\pm U_j^\pm \quad \text{for } |j - k| > 1. \end{aligned} \tag{3.13}$$

Let us now show commutation relations (3.10), making use of the Temperley-Lieb algebra. The operator $\check{R}_{j,j+1}^+(u)$ is decomposed in terms of the generators of the Temperley-Lieb algebra as follows [36].

$$\check{R}_{j,j+1}^+(u) = I - b(u)U_j^+. \tag{3.14}$$

Lemma 8. *The monodromy matrix of the six-vertex model is expressed in terms of the generators of the Temperley-Lieb algebra as follows.*

$$\begin{aligned} &\check{R}_{L-1,L}^+(\lambda - \xi_L) \cdots \check{R}_{1,2}^+(\lambda - \xi_2) \check{R}_{0,1}^+(\lambda - \xi_1) \\ &= \sum_{k=0}^L (-1)^k \sum_{0 \leq i_1 < \dots < i_k < L} \left(\prod_{j=1}^k b(\lambda - \xi_{i_j}) \right) U_{i_k}^+ \cdots U_{i_2}^+ U_{i_1}^+. \end{aligned} \tag{3.15}$$

Lemma 9. *The generators U_j^+ commute with the generators of $U_q(sl_2)$. For $x = X^\pm, K$ and for $j = 0, 1, \dots, L-1$, we have in the tensor-product representation*

$$[U_j^+, \Delta^{(L)}(x)] = 0. \tag{3.16}$$

Proof of proposition 7. From lemmas 8 and 9 we have lemma 6, which is equivalent to proposition 7.

We now show that in the limit of taking u to $-\infty$, $\hat{R}^+(u)$ is equivalent to the spin-1/2 matrix representation of the universal R -matrix \mathcal{R} of $U_q(sl(2))$. An explicit expression of \mathcal{R} is given by

$$\mathcal{R} = q^{-\frac{1}{2}H \otimes H} \exp_q(-(q - q^{-1})K^{-1}X^+ \otimes X^-K) \tag{3.17}$$

where $\exp_q x$ denotes the following series:

$$\exp_q x = \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{[n]_q!} x^n. \tag{3.18}$$

Here q is generic. We recall that $[n]_q$ denotes the q -integer of an integer n : $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. Putting $X^+ = e^{1,2}$, $X^- = e^{2,1}$ and $K = \text{diag}(q, q^{-1})$ in the series (3.17), we have the following matrix representation.

$$R_{\frac{1}{2},\frac{1}{2}} = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & q^{1/2} & -q^{1/2}(q - q^{-1}) & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}. \tag{3.19}$$

Thus, for arbitrary u , we have the following:

$$R_{12}^+(u) = c^+(u)\Pi_{12} + q^{-1/2}b(u)R_{\frac{1}{2},\frac{1}{2}} \quad (3.20)$$

Therefore, we have $R^+(-\infty) = q^{1/2}R_{\frac{1}{2},\frac{1}{2}}$.

We remark that some relations equivalent to (3.16) have been shown in association with the $sl(2)$ loop algebra symmetry of the XXZ spin chain at roots of unity [37].

3.3 Gauge transformations

Let us introduce operators Φ_j with arbitrary parameters ϕ_j for $j = 0, 1, \dots, L$ as follows:

$$\Phi_j = \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi_j} \end{pmatrix}_{[j]} = I^{\otimes(j)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi_j} \end{pmatrix} \otimes I^{\otimes(L-j)}. \quad (3.21)$$

In terms of $\chi_{jk} = \Phi_j \Phi_k$, we define a similarity transformation on the R -matrix by

$$R_{jk}^\chi = \chi_{jk} R_{jk} \chi_{jk}^{-1} \quad (3.22)$$

Explicitly, the following two matrix elements are transformed.

$$\left(R_{jk}^\chi\right)_{12}^{21} = c(\lambda_j, \lambda_k) e^{\phi_j - \phi_k}, \quad \left(R_{jk}^\chi\right)_{21}^{12} = c(\lambda_j, \lambda_k) e^{-\phi_j + \phi_k}. \quad (3.23)$$

We now put $\phi_j = \lambda_j$ in eq. (3.21) for $j = 0, 1, \dots, L$. For $j, k = 0, 1, \dots, L$, we have

$$R_{jk}^\pm(\lambda_j, \lambda_k) = (\chi_{jk})^{\pm 1} R_{jk}(\lambda_j, \lambda_k) (\chi_{jk})^{\mp 1}. \quad (3.24)$$

Thus, the asymmetric R -matrices $R_{12}^\pm(\lambda_1, \lambda_2)$ are derived from the symmetric one through the gauge transformation χ_{jk} .

For the monodromy matrix, in terms of the inhomogeneous parameters, ξ_1, \dots, ξ_L , we put $\lambda_j = \xi_j$ for $j = 1, \dots, L$. We define $\chi_{012\dots L}$ by $\chi_{012\dots L} = \Phi_0 \Phi_1 \cdots \Phi_L$. Then, the asymmetric monodromy matrices are transformed into the symmetric one as follows.

$$R_{0,12\dots L}^\pm = (\chi_{012\dots L})^{\pm 1} R_{0,12\dots L} (\chi_{012\dots L})^{\mp 1}. \quad (3.25)$$

We note that the asymmetric R -matrices $\check{R}_{j,j+1}^\pm(u)$ are derived from the symmetric R -matrix through the gauge transformations, and they are related to the Jones polynomial. [21]

3.4 Affine quantum group symmetry

The affine quantum algebra $U_q(\widehat{sl}_2)$ is an associative algebra over \mathbf{C} generated by X_i^\pm, K_i^\pm for $i = 0, 1$ with the following relations:

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, \quad K_i X_i^\pm K_i^{-1} = q^{\pm 2} X_i^\pm, \quad K_i X_j^\pm K_i^{-1} = q^{\mp 2} X_j^\pm \quad (i \neq j), \\ [X_i^+, X_j^-] &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ (X_i^\pm)^3 X_j^\pm &- [3]_q (X_i^\pm)^2 X_j^\pm X_i^\pm + [3]_q X_i^\pm X_j^\pm (X_i^\pm)^2 - X_j^\pm (X_i^\pm)^3 = 0 \quad (i \neq j). \end{aligned} \quad (3.26)$$

The algebra $U_q(\widehat{sl}_2)$ is also a Hopf algebra over \mathbf{C} with comultiplication

$$\begin{aligned}\Delta(X_i^+) &= X_i^+ \otimes 1 + K_i \otimes X_i^+, \quad \Delta(X_i^-) = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-, \\ \Delta(K_i) &= K_i \otimes K_i,\end{aligned}\tag{3.27}$$

and antipode: $S(K_i) = K_i^{-1}$, $S(X_i) = -K_i^{-1}X_i^+$, $S(X_i^-) = -X_i^-K_i$.

We now introduce evaluation representations for $U_q(\widehat{sl}_2)$ [22]. For a given complex number a there is a homomorphism of algebras $\varphi_a: U_q(\widehat{sl}_2) \rightarrow U_q(sl_2)$ such that

$$\begin{aligned}\varphi_a(X_0^\pm) &= \exp(\pm a) X^\mp, \quad \varphi_a(K_0) = K^{-1}, \\ \varphi_a(X_1^\pm) &= X^\pm, \quad \varphi_a(K_1) = K.\end{aligned}\tag{3.28}$$

We denote by (π, V) a representation of an algebra \mathcal{A} such that $\pi(x)$ give linear maps on vector space V for $x \in \mathcal{A}$. For a given finite-dimensional representation (π_V, V) of $U_q(sl_2)$ we have a finite-dimensional representation $(\pi_{V(a)}, V(a))$ of $U_q(\widehat{sl}_2)$ through homomorphism φ_a , i.e. $\pi_{V(a)}(x) = \pi_V(\varphi_a(x))$ for $x \in U_q(\widehat{sl}_2)$. We call $(\pi_{V(a)}, V(a))$ or $V(a)$ the *evaluation representation* of V and nonzero parameter a the *evaluation parameter* of $V(a)$. If V is $(2s+1)$ -dimensional, then we also denote it by $V^{(2s)}(a)$. Hereafter we express $2s$ by an integer ℓ .

Similarly as (3.14), we have the following decomposition:

$$\check{R}_{j,j+1}^-(u) = I - b(u)U_j^-, \quad \text{for } j = 0, 1, \dots, L-1.\tag{3.29}$$

Lemma 10. *Generators U_j^- commute with X_0^\pm and K_0 of $U_q(\widehat{sl}_2)$ in the tensor-product representation $V^{(1)}(a_0) \otimes \dots \otimes V^{(1)}(a_L)$ with $a_0 = a_1 = \dots = a_L = a$. For $j = 0, 1, \dots, L-1$, we have*

$$[U_j^-, \varphi_a^{\otimes(L+1)}(\Delta^{(L)}(x))] = 0 \quad (x = X_0^\pm, K_0).\tag{3.30}$$

Proof. Let us denote by U^- the 4×4 matrix given by the decomposition: $\check{R}(u) = I - b(u)U^-$. Through an explicit calculation we show that U^- and $\varphi_a \otimes \varphi_b (\Delta(X_0^\pm))$ commute if $a = b$:

$$[U^-, \varphi_a \otimes \varphi_a (\Delta(X_0^\pm))] = 0.\tag{3.31}$$

We derive (3.30) through (3.31). \square

In the spin-1/2 representation of $U_q(sl_2)$, we thus have the following relations:

$$R_{12}^-(u) \varphi_a^{\otimes 2} \Delta(x) = \varphi_a^{\otimes 2} (\tau \circ \Delta(x)) R_{12}^-(u) \quad \text{for } x = X_0^\pm, K_0.\tag{3.32}$$

Here we note that u is arbitrary and independent of X_0^\pm, K_0 .

Similarly as lemmas 8 and 9, we have from lemma 10 the quantum-group symmetry of the monodromy matrix $R_{0,12\dots L}^-$:

Proposition 11. Let σ_c be a cyclic permutation: $\sigma_c = (01 \cdots L)$. In the evaluation representation (3.28) we have, for $x = X_0^\pm, K_0$, the following:

$$R_{0,12 \cdots L}^-(\lambda; \xi_1, \dots, \xi_L) \varphi_a^{\otimes(L+1)} \left(\Delta^{(L)}(x) \right) = \varphi_a^{\otimes(L+1)} \left(\sigma_c \circ \Delta^{(L)}(x) \right) R_{0,12 \cdots L}^-(\lambda; \xi_1, \dots, \xi_L). \quad (3.33)$$

Here parameters $\lambda, \xi_1, \dots, \xi_L$ are arbitrary and independent of $x = X_0^\pm, K_0$.

Let us now make a summary of the symmetry relations of R_{12}^+ . Here we recall that $R_{12}^+(\lambda_1, \lambda_2) \in \text{End}(V(\lambda_1) \otimes V(\lambda_2))$. For simplicity, we put $a = 0$ in (3.33). Combining (3.8) and (3.32) We have the following relations:

$$\begin{aligned} R_{12}^+(\lambda_1, \lambda_2) \varphi_0^{\otimes 2} \left(\Delta(X_1^\pm) \right) &= \varphi_0^{\otimes 2} \left(\tau \circ \Delta(X_1^\pm) \right) R_{12}^+(\lambda_1, \lambda_2), \\ R_{12}^+(\lambda_1, \lambda_2) (\chi_{12})^2 \varphi_0^{\otimes 2} \left(\Delta(X_0^\pm) \right) (\chi_{12})^{-2} &= (\chi_{12})^2 \varphi_0^{\otimes 2} \left(\tau \circ \Delta(X_0^\pm) \right) (\chi_{12})^{-2} R_{12}^+(\lambda_1, \lambda_2), \\ R_{12}^+(\lambda_1, \lambda_2) \varphi_0^{\otimes 2} \left(\Delta(K_i^\pm) \right) &= \varphi_0^{\otimes 2} \left(\tau \circ \Delta(K_i^\pm) \right) R_{12}^+(\lambda_1, \lambda_2) \text{ for } i = 0, 1. \end{aligned} \quad (3.34)$$

Let us now consider $\varphi_{a_1} \otimes \varphi_{a_2}$ with $a_j = 2\lambda_j$ for $j = 1, 2$. We have

$$\varphi_{2\lambda_1} \otimes \varphi_{2\lambda_2} \left(\Delta(X_0^\pm) \right) = (\chi_{12})^2 \varphi_0 \otimes \varphi_0 \left(\Delta(X_0^\pm) \right) (\chi_{12})^{-2} \quad (3.35)$$

Thus, relations (3.34) are now expressed as follows.

$$R_{12}^+(\lambda_1, \lambda_2) \varphi_{2\lambda_1} \otimes \varphi_{2\lambda_2} \left(\Delta(x) \right) = \varphi_{2\lambda_1} \otimes \varphi_{2\lambda_2} \left(\tau \circ \Delta(x) \right) R_{12}^+(\lambda_1, \lambda_2), \text{ for } x = X_0^\pm, X_1^\pm, K_0, K_1. \quad (3.36)$$

In (3.36) all the parameters are now associated with the evaluation parameters of the tensor product $V(2\lambda_1) \otimes V(2\lambda_2)$. Therefore, we conclude that the asymmetric R -matrix $R_{12}^+(\lambda_1, \lambda_2)$ satisfies the affine quantum-group symmetry.

The fundamental commutation relations (3.10) and (3.33) are summarized as follows.

Proposition 12. Let σ_c be a cyclic permutation: $\sigma_c = (01 \cdots L)$. The asymmetric R -matrix R^+ satisfies the commutation relations for the affine-quantum group:

$$R_{0,12 \cdots L}^+(\lambda_0) \left(\Delta^{(n)}(x) \right)_{01 \cdots L} = \left(\sigma_c \circ \Delta^{(L)}(x) \right)_{01 \cdots L} R_{0,12 \cdots L}^+(\lambda_0) \quad \text{for all } x \in U_q(\widehat{sl}_2). \quad (3.37)$$

Here the symbol $(x)_{01 \cdots n}$ denotes the matrix representation of x in the tensor product of evaluation representations, $V(2\lambda_0) \otimes V(2\xi_1) \otimes \cdots \otimes V(2\xi_L)$.

3.5 Symmetry relations of $U_q(\widehat{sl}_2)$ for all permutations

Let us generalize relations (3.37). Making an extensive use of relations (3.36), we can show commutation relations for R_p^σ for all permutations σ . In fact, we can prove (3.37) also by the method for showing proposition A.2. In Appendix A we shall show in proposition A.2 how we generalize the symmetry relation of R_{12} such as (3.36) into those of R_p^σ for any permutation σ .

We now formulate the symmetry relations in terms of the symmetric R -matrices. Let us denote by $\bar{\chi}$ the inverse of the gauge transformation χ . We express by $(\Delta^{(n)}(x))_{01\dots n}^{\bar{\chi}}$ the following:

$$(\Delta^{(n)}(x))_{01\dots n}^{\bar{\chi}} = (\chi_{01\dots n})^{-1} \varphi_{2\lambda_0} \otimes \varphi_{2\xi_1} \otimes \dots \otimes \varphi_{2\xi_n} (\Delta^{(n)}(x)) \chi_{01\dots n} \quad (3.38)$$

Proposition 13. *Let p_q be an increasing sequence of $n+1$ integers: $p_q = (0, 1, 2, \dots, n)$, and σ a permutation on $n+1$ integers, $0, 1, \dots, n$. With the symmetric R -matrices, we have*

$$R_{p_q}^\sigma(\lambda_0) (\Delta^{(n)}(x))_{01\dots n}^{\bar{\chi}} = (\sigma \circ \Delta^{(n)}(x))_{01\dots n}^{\bar{\chi}} R_{p_q}^\sigma(\lambda_0) \quad \text{for all } x \in U_q(\widehat{sl}_2). \quad (3.39)$$

4 Projection operators and the fusion procedure

4.1 Projection operators

Let us recall that $\check{R}_{12}^+(u)$ has been defined by

$$\check{R}_{12}^+(u) = \Pi_{12} R_{12}^+(u).$$

We define operator P_{12}^+ by

$$P_{12}^+ = \check{R}_{12}^+(\eta) \quad (4.1)$$

Explicitly we have

$$P_{12}^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q}{[2]} & \frac{1}{[2]} & 0 \\ 0 & \frac{1}{[2]} & \frac{q}{[2]} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[12]} . \quad (4.2)$$

By making use of the matrix representation (4.2) it is easy to show that operator P_{12}^+ is idempotent:

$$(P_{12}^+)^2 = P_{12}^+ \quad (4.3)$$

Thus, we may consider that the operator P_{12}^+ is a projection operator. In fact, P_{12}^+ is nothing but the q -analogue of the Young operator which projects out the spin-1 representation of $U_q(sl_2)$ from of the tensor product of two spin-1/2 representations. It should be noticed that in the case of symmetric R -matrix, $R(\eta)^2$ is not equal to $R(\eta)$.

We now introduce projection operators for the spin- s irreducible representations of $U_q(sl_2)$. Hereafter we set $\ell = 2s$. We define projection operator $P_{12\dots\ell}^{(\ell)}$ acting on the tensor product $V_{12\dots\ell}$ of spin-1/2 representations V , i.e. $V_{12\dots\ell} = V^{\otimes\ell}$, by the following recursive relations:

$$P_{12\dots\ell}^{(\ell)} = P_{12\dots\ell-1}^{(\ell-1)} \check{R}_{\ell-1,\ell}^+((\ell-1)\eta) P_{12\dots\ell-1}^{(\ell-1)} \quad (4.4)$$

Here $P_{12}^{(1)} = P_{12}^+$.

Making use of the Yang-Baxter equations (2.26) and through induction on ℓ we can show

$$\left(P_{12\cdots\ell}^{(\ell)} \right)^2 = P_{12\cdots\ell}^{(\ell)} \quad (4.5)$$

Thus, operator $P_{12\cdots\ell}^{(\ell)}$ gives a projection operator. Similarly, we define projection operators acting on $V_{j,j+1\cdots j+\ell-1}$ recursively by

$$P_{j,j+1\cdots j+\ell-1}^{(\ell)} = P_{j,j+1\cdots j+\ell-2}^{(\ell-1)} \check{R}_{j+\ell-2,j+\ell-1}^+((\ell-1)\eta) P_{j,j+1\cdots j+\ell-2}^{(\ell-1)} \quad (4.6)$$

Hereafter we shall abbreviate $P_{j,j+1\cdots j+\ell-1}^{(\ell)}$ by $P_j^{(\ell)}$.

From idempotency (4.5) and by the Yang-Baxter equations we can show the following:

Lemma 14. Suppose that inhomogeneous parameters $\xi_j, \xi_{j+1}, \dots, \xi_{j+\ell-1}$ are given by $\xi_{j+i-1} = z - (i-1)\eta$ for $i = 1, 2, \dots, \ell$ with a constant z . Then, the monodromy matrix $R_{0,12\cdots L}^+$ satisfies the following property:

$$P_j^{(\ell)} R_{0,12\cdots L}^+ = P_j^{(\ell)} R_{0,12\cdots L}^+ P_j^{(\ell)}. \quad (4.7)$$

Making use of (2.28) we can express projection operators in terms of R -matrices.

$$P_1^{(\ell)} = \left(\prod_{j=1}^{\ell} \Pi^{(j \ \ell-j+1)} \right) R_{\ell-1,\ell}^+ \cdots R_{2,3\cdots\ell}^+ R_{1,2\cdots\ell}^+ \quad (4.8)$$

4.2 Fusion of monodromy matrices

4.2.1 The case of tensor product of spin- s representations

We first consider the case of tensor product of spin- s representations. We set $L = N_s \ell$. We introduce a set of parameters, $\xi_1^{(\ell)}, \xi_2^{(\ell)}, \dots, \xi_L^{(\ell)}$, as follows:

$$\xi_{(k-1)\ell+j}^{(\ell)} = \zeta_k - (j-1)\eta + (\ell-1)\eta/2 \quad \text{for } j = 1, \dots, \ell, \text{ and } k = 1, \dots, N_s. \quad (4.9)$$

Let us set inhomogeneous parameters $\xi_1, \xi_2, \dots, \xi_L$, by $\xi_j = \xi_j^{(\ell)}$ for $j = 1, 2, \dots, L$. We define the monodromy matrix $T_0^{(\ell+)}(u; \zeta_1, \dots, \zeta_{N_s})$ acting on the tensor product of spin- s representations, $V^{(2s)}(\zeta_1) \otimes V^{(2s)}(\zeta_{N_s})$ by

$$T_0^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_{N_s}) = \prod_{k=0}^{N_s-1} P_{\ell k+1}^{(\ell)} \cdot R_{0,12\cdots L}^+(\lambda_0; \xi_1^{(\ell)}, \dots, \xi_L^{(\ell)}) \cdot \prod_{k=0}^{N_s-1} P_{\ell k+1}^{(\ell)} \quad (4.10)$$

Making use of properties of projection operators (4.5) and (4.7), we can show the Yang-Baxter equation [?, 19]

$$\begin{aligned} & R_{ab}^+(\lambda_a - \lambda_b) T_a^{(\ell+)}(\lambda_a; \zeta_1, \dots, \zeta_{N_s}) T_b^{(\ell+)}(\lambda_b; \zeta_1, \dots, \zeta_{N_s}) \\ &= T_b^{(\ell+)}(\lambda_b; \zeta_1, \dots, \zeta_{N_s}) T_a^{(\ell+)}(\lambda_a; \zeta_1, \dots, \zeta_{N_s}) R_{ab}^+(\lambda_a - \lambda_b) \end{aligned} \quad (4.11)$$

Through the inverse of the gauge transformation, we derive the symmetric spin- s monodromy matrix as follows:

$$T_0^{(\ell)}(\lambda_0; \zeta_1, \dots, \zeta_{N_s}) = \prod_{k=0}^{N_s-1} \left(P_{\ell k+1}^{(\ell)} \right)^{\bar{\chi}} \cdot R_{0,12\dots L}(\lambda_0; \xi_1^{(\ell)}, \dots, \xi_L^{(\ell)}) \cdot \prod_{k=0}^{N_s-1} \left(P_{\ell k+1}^{(\ell)} \right)^{\bar{\chi}} \quad (4.12)$$

where we have defined the transformed projectors by

$$\left(P_{\ell k+1}^{(\ell)} \right)^{\bar{\chi}} = (\chi_{01\dots L})^{-1} P_{\ell k+1}^{(\ell)} (\chi_{01\dots L}) \quad (4.13)$$

4.2.2 The case of mixed spins

Let us consider the tensor product of representations with different spins, s_1, s_2, \dots, s_r . Here we introduce ℓ_j by $\ell_j = 2s_j$ for $j = 1, 2, \dots, r$, and we assume that $\ell_1 + \ell_2 + \dots + \ell_r = L$. @ Let us introduce a set of parameters, $\xi_1^{(\ell)}, \xi_2^{(\ell)}, \dots, \xi_L^{(\ell)}$, as follows:

$$\xi_{\ell_1+\dots+\ell_{k-1}+j}^{(\ell)} = \zeta_k - (j-1)\eta + (\ell_k-1)\eta/2 \quad \text{for } j = 1, \dots, \ell_k, \text{ and } k = 1, \dots, r. \quad (4.14)$$

We define the asymmetric monodromy matrix for the mixed spin case $T_0^{(\ell+)}(u; \zeta_1, \dots, \zeta_r)$ acting on the tensor product representation $V_{\ell_1}(\zeta_1) \otimes \dots \otimes V_{\ell_r}(\zeta_r)$ by

$$T_0^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_r) = \prod_{k=1}^r P_{\ell_1+\dots+\ell_{k-1}+1}^{(\ell_k)} \cdot R_{0,12\dots L}(\lambda_0; \xi_1^{(\ell)}, \dots, \xi_L^{(\ell)}) \cdot \prod_{k=1}^r P_{\ell_1+\dots+\ell_{k-1}+1}^{(\ell_k)}. \quad (4.15)$$

It is easy to show that they satisfy the Yang-Baxter equations.

$$\begin{aligned} & R_{ab}^+(\lambda_a - \lambda_b) T_a^{(\ell+)}(\lambda_a; \zeta_1, \dots, \zeta_{N_s}) T_b^{(\ell+)}(\lambda_b; \zeta_1, \dots, \zeta_{N_s}) \\ &= T_b^{(\ell+)}(\lambda_b; \zeta_1, \dots, \zeta_{N_s}) T_a^{(\ell+)}(\lambda_a; \zeta_1, \dots, \zeta_{N_s}) R_{ab}^+(\lambda_a - \lambda_b) \end{aligned} \quad (4.16)$$

We also define the symmetric monodromy matrix for the mixed spin case $T_0^{(\ell)}(u; \zeta_1, \dots, \zeta_r)$ as follows.

$$T_0^{(\ell)}(\lambda_0; \zeta_1, \dots, \zeta_r) = \prod_{k=1}^r \left(P_{\ell_1+\dots+\ell_{k-1}+1}^{(\ell_k)} \right)^{\bar{\chi}} \cdot R_{0,12\dots L}(\lambda_0; \xi_1^{(\ell)}, \dots, \xi_L^{(\ell)}) \cdot \prod_{k=1}^r \left(P_{\ell_1+\dots+\ell_{k-1}+1}^{(\ell_k)} \right)^{\bar{\chi}} \quad (4.17)$$

4.3 Higher-spin L -operators

We now define the basis vectors of the $(\ell+1)$ -dimensional irreducible representation of $U_q(sl_2)$, $||\ell, n\rangle$ for $n = 0, 1, \dots, \ell$ as follows. We define $||\ell, 0\rangle$ by

$$||\ell, 0\rangle = |1\rangle_1 \otimes |1\rangle_2 \otimes \dots \otimes |1\rangle_\ell \quad (4.18)$$

Here $|\alpha\rangle_j$ for $\alpha = 1, 2$ denote the basis vectors of the spin-1/2 representation defined on the j th position in the tensor product. We define $||\ell, n\rangle$ for $n \geq 1$ by

$$||\ell, n\rangle = \left(\Delta^{(\ell-1)}(X^-) \right)^n ||\ell, 0\rangle \frac{1}{[n]_q!}. \quad (4.19)$$

Then we have

$$|\ell, n\rangle = \sum_{1 \leq i_1 < \dots < i_n \leq \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- |0\rangle q^{i_1+i_2+\dots+i_n-n\ell+n(n-1)/2} \quad (4.20)$$

It is easy to show the following:

$$P_{12\dots\ell}^{(\ell)} |\ell, n\rangle = |\ell, n\rangle \quad (4.21)$$

We define the conjugate vectors by the following conditions:

$$\langle \ell, n| P_{12\dots\ell}^{(\ell)} = \langle \ell, n| \quad (4.22)$$

with the normalization condition: $\langle \ell, n| |\ell, n\rangle = 1$. Let us define the q -factorial, $[n]_q!$, by

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q . \quad (4.23)$$

For integers m and n satisfying $m \geq n$ we define the q -binomial coefficients as follows

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[m-n]_q! [n]_q!} \quad (4.24)$$

Then we have the following expression of the conjugate vectors

$$\langle \ell, n| = \left[\begin{array}{c} \ell \\ n \end{array} \right]_q^{-1} q^{n(\ell-n)} \sum_{1 \leq i_1 < \dots < i_n \leq \ell} \langle 0| \sigma_{i_1}^+ \cdots \sigma_{i_n}^+ q^{i_1+\dots+i_n-n\ell+n(n-1)/2} \quad (4.25)$$

The projection operators are given explicitly as follows.

$$P_{12\dots\ell}^{(\ell)} = \sum_{n=0}^{\ell} |\ell, n\rangle \langle \ell, n| \quad (4.26)$$

We define the L -operator of the spin- $\ell/2$ XXZ model by

$$L^{(\ell+)}(\lambda_0) = P_1^{(\ell)} R_{0,12\dots\ell}^+ P_1^{(\ell)} . \quad (4.27)$$

and then by the inverse gauge transformation we have

$$L^{(\ell)}(\lambda_0) = \left(P_1^{(\ell)} \right)^{\bar{\chi}} R_{0,12\dots\ell} \left(P_1^{(\ell)} \right)^{\bar{\chi}} . \quad (4.28)$$

Let us define $|\ell, n\rangle$ and their conjugates $\langle \ell, n|$ by

$$\begin{aligned} |\ell, n\rangle &= N(\ell, n) |\ell, n\rangle \\ \langle \ell, n| &= \langle \ell, n| \frac{1}{N(\ell, n)} . \end{aligned} \quad (4.29)$$

The matrix elements of the L -operator are given by

$$\langle \ell, a| L^{(\ell)}(\lambda) |\ell, b\rangle = \begin{pmatrix} \langle \ell, a| L_{11}^{(\ell)}(\lambda) |\ell, b\rangle & \langle \ell, a| L_{12}^{(\ell)}(\lambda) |\ell, b\rangle \\ \langle \ell, a| L_{21}^{(\ell)}(\lambda) |\ell, b\rangle & \langle \ell, a| L_{22}^{(\ell)}(\lambda) |\ell, b\rangle \end{pmatrix}_{[0]} \quad (4.30)$$

for $a, b = 0, 1, \dots, \ell$. Choosing the normalization factors $N(\ell, n)$, we can derive the following symmetric expression of the L -operator:

$$L^{(\ell)}(\lambda) = \frac{1}{2 \sinh(u + \ell\eta/2)} \begin{pmatrix} zK^{1/2} - z^{-1}K^{-1/2} & 2 \sinh \eta X^- \\ 2 \sinh \eta X^+ & zK^{-1/2} - z^{-1}K^{1/2} \end{pmatrix}_{[0]} \quad (4.31)$$

Here X^\pm and K are in the $(\ell+1)$ -dimensional representation of $U_q(sl_2)$, and $u = \lambda - \xi_1 + \ell\eta/2$ and $z = \exp u$. Explicitly they are given by

$$\begin{aligned} \langle \ell, a | X^+ | \ell, b \rangle &= \delta_{a,b-1} [\ell - a]_q, \\ \langle \ell, a | X^- | \ell, b \rangle &= \delta_{a,b+1} [a]_q, \\ \langle \ell, a | K | \ell, b \rangle &= \delta_{a,b} q^{\ell-2a} \quad \text{for } a, b = 0, 1, \dots, \ell. \end{aligned} \quad (4.32)$$

4.4 Algebraic Bethe-ansatz method for higher-spin cases

We now discuss the eigenvalues of the transfer matrix of an integrable higher-spin XXZ spin chain constructed by the fusion method. We consider the case of mixed spins, where we define the transfer matrix on the tensor product of spin- s_j representations for $j = 1, 2, \dots, r$.

We define A , B , C , and D operators of the algebraic Bethe ansatz for higher-spin cases by the following matrix elements of the monodromy matrix:

$$\begin{pmatrix} A^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_r) & B^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_r) \\ C^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_r) & D^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_r) \end{pmatrix} = T_0^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_r). \quad (4.33)$$

In terms of projection operators we have

$$B_{1\dots r}^{(\ell+)}(\lambda_0; \zeta_1, \dots, \zeta_r) = \prod_{k=1}^r P_{\ell(k-1)+1}^{(\ell_k)} \cdot B_{12\dots L}^+(\lambda_0; \xi_1^{(\ell)}, \dots, \xi_L^{(\ell)}) \prod_{k=1}^r P_{\ell(k-1)+1}^{(\ell_k)}. \quad (4.34)$$

We define $A^{(\ell)}$, $B^{(\ell)}$, $C^{(\ell)}$ and $D^{(\ell)}$ similarly for the monodromy matrix $T_0^{(\ell)}(\lambda_0; \zeta_1, \dots, \zeta_r)$.

The operators $A^{(\ell+)}$'s of the asymmetric monodromy matrix $R_{0,12\dots n}^+$ are related to the symmetric ones $A^{(\ell)}$'s as follows.

$$\begin{aligned} R_{0,12\dots n}^+(\lambda_0, \{\zeta_i\}) &= \begin{pmatrix} A^{(\ell+)}(\lambda_0; \{\zeta_i\}) & B^{(\ell+)}(\lambda_0; \{\zeta_i\}) \\ C^{(\ell+)}(\lambda_0; \{\zeta_i\}) & D^{(\ell+)}(\lambda_0; \{\zeta_i\}) \end{pmatrix} \\ &= \begin{pmatrix} \chi_{12\dots L} A^{(\ell)}(\lambda_0; \{\zeta_i\}) \chi_{12\dots L}^{-1} & e^{-\lambda_0} \chi_{12\dots L} B^{(\ell)}(\lambda_0; \{\zeta_i\}) \chi_{12\dots L}^{-1} \\ e^{\lambda_0} \chi_{12\dots L} C^{(\ell)}(\lambda_0; \{\zeta_i\}) \chi_{12\dots L}^{-1} & \chi_{12\dots L} D^{(\ell)}(\lambda_0; \{\zeta_i\}) \chi_{12\dots L}^{-1} \end{pmatrix} \end{aligned} \quad (4.35)$$

It follows from the Yang-Baxter equations (4.16) that the A, B, C, D operators in the higher-spin case also satisfy the standard commutation relations.

$$A^{(\ell+)}(\lambda_1) B^{(\ell+)}(\lambda_2) = \frac{1}{b(\lambda_2 - \lambda_1)} B^{(\ell+)}(\lambda_2) A^{(\ell+)}(\lambda_1) - \frac{c^-(\lambda_2 - \lambda_1)}{b(\lambda_2 - \lambda_1)} B^{(\ell+)}(\lambda_1) A^{(\ell+)}(\lambda_2) \quad (4.36)$$

Through the inverse gauge transformation $\bar{\chi}$ we have

$$A^{(\ell)}(\lambda_1)B^{(\ell)}(\lambda_2) = \frac{1}{b(\lambda_2 - \lambda_1)}B^{(\ell)}(\lambda_2)A^{(\ell)}(\lambda_1) - \frac{c(\lambda_2 - \lambda_1)}{b(\lambda_2 - \lambda_1)}B^{(\ell)}(\lambda_1)A^{(\ell)}(\lambda_2) \quad (4.37)$$

Therefore, we derive Bethe ansatz eigenvectors of the higher-spin transfer matrix by the same method as the case of spin-1/2.

Let us denote by $|0\rangle$ the vacuum state where all spins are up. Noting

$$\prod_{k=1}^r P_{\ell(k-1)+1}^{(\ell_k)}|0\rangle = |0\rangle, \quad (4.38)$$

it is easy to show the following relations:

$$\begin{aligned} A^{(\ell)}(\lambda)|0\rangle &= a^{(\ell)}(\lambda; \{\zeta_k\})|0\rangle, \\ D^{(\ell)}(\lambda)|0\rangle &= d^{(\ell)}(\lambda; \{\zeta_k\})|0\rangle, \end{aligned} \quad (4.39)$$

where $a^{(\ell)}(\lambda; \{\zeta_k\})$ and $d^{(\ell)}(\lambda; \{\zeta_k\})$ are given by

$$\begin{aligned} a^{(\ell)}(\lambda; \{\zeta_k\}) &= a^{(\ell)}(\lambda; \{\xi_j^{(\ell)}\}) = 1, \\ d^{(\ell)}(\lambda; \{\zeta_k\}) &= d^{(\ell)}(\lambda; \{\xi_j^{(\ell)}\}) = \prod_{j=1}^L b(\lambda - \xi_j^{(\ell)}) = \prod_{k=1}^r \frac{\sinh(\lambda - \zeta_k - (\ell_k - 1)\eta/2)}{\sinh(\lambda - \zeta_k + (\ell_k + 1)\eta/2)} \end{aligned} \quad (4.40)$$

Thus, the vector $B^{(\ell)}(\lambda_1) \cdots B^{(\ell)}(\lambda_n)|0\rangle$ becomes an eigenvector of the transfer matrix $A^{(\ell)}(\lambda) + D^{(\ell)}(\lambda)$ with the following eigenvalue

$$\tau^{(\ell)}(\mu) = \prod_{j=1}^n \frac{\sinh(\lambda_j - \mu + \eta)}{\sinh(\lambda_j - \mu)} + \prod_{k=1}^r \frac{\sinh(\mu - \zeta_k - (\ell_k - 1)\eta/2)}{\sinh(\mu - \zeta_k + (\ell_k + 1)\eta/2)} \cdot \prod_{j=1}^n \frac{\sinh(\mu - \lambda_j + \eta)}{\sinh(\mu - \lambda_j)} \quad (4.41)$$

if rapidities $\tilde{\lambda}_j = \lambda_j + \eta/2$ satisfy the Bethe ansatz equations

$$\prod_{k=1}^r \frac{\sinh(\tilde{\lambda}_\alpha - \zeta_k + \ell_k \eta/2)}{\sinh(\tilde{\lambda}_\alpha - \zeta_k - \ell_k \eta/2)} = \prod_{\beta=1; \beta \neq \alpha}^n \frac{\sinh(\tilde{\lambda}_\alpha - \tilde{\lambda}_\beta + \eta)}{\sinh(\tilde{\lambda}_\alpha - \tilde{\lambda}_\beta - \eta)} \quad (4.42)$$

5 Pseudo-diagonalization of the B and C operators

5.1 Diagonalizing the A and D operators

5.1.1 The F -basis

In order to formulate the derivation of the pseudo-diagonalized forms of B and C operators for the XXZ case, we briefly formulate some symbols and review some useful formulas shown in Ref. [6] in SS5.1. First, we introduce the F -basis .

Definition 15. (*Partial F and total F*) We define partial F by

$$\begin{aligned} F_{1,2\cdots n} &= e_1^{11} + e_1^{22} R_{1,2\cdots n} \\ F_{12\cdots n-1,n} &= e_n^{22} + e_n^{11} R_{12\cdots n-1,n} \end{aligned} \quad (5.1)$$

We define total F recursively with respect to n by

$$F_{12\cdots n} = F_{12\cdots n-1} F_{12\cdots n-1,n} \quad (5.2)$$

Lemma 16. (*Cocycle conditions*)

$$\begin{aligned} F_{1,2} F_{12,3} &= F_{23} F_{1,23} \\ F_{1,2\cdots n-1} F_{12\cdots n-1,n} &= F_{2\cdots n-1,n} F_{1,2\cdots n} \end{aligned} \quad (5.3)$$

Proof. Expressing the F-basis in terms of R-matrices through (5.1), we show that the cocycle conditions of the F-basis are reduced to those of the R-matrices, which are shown in Appendix B. \square

From the cocycle conditions we have the following:

Lemma 17.

$$F_{12\cdots n} = F_{2\cdots n} F_{1,2\cdots n} \quad (5.4)$$

5.1.2 Basic properties of the R-matrix

Let us introduce some important properties of the R-matrix of the XXZ spin-chain.

The R-matrix is invariant under the charge conjugation. For the symmetric R-matrix, we define the charge conjugation operator \mathcal{C} by

$$\mathcal{C}_{12\cdots n} = \sigma_1^x \cdots \sigma_n^x \quad (5.5)$$

For a given operator $A \in End(V(\lambda_1) \otimes \cdots \otimes V(\lambda_n))$ we define \bar{A} by $\bar{A} = \mathcal{C}_{1\cdots n} A \mathcal{C}_{1\cdots n}$. For instance, we define $\bar{F}_{0,1\cdots n}$ by

$$\bar{F}_{0,1\cdots n} = \mathcal{C}_{01\cdots n} F_{0,1\cdots n} \mathcal{C}_{01\cdots n} \quad (5.6)$$

Proposition 18. *The charge conjugation operator \mathcal{C} commutes with the monodromy matrix of the symmetric R-matrix:*

$$[\mathcal{C}_{01\cdots n}, R_{0,1\cdots n}] = 0 \quad (5.7)$$

We thus have $\bar{A}_{1\cdots n}(\lambda_0) = D_{1\cdots n}(\lambda_0)$ and $\bar{B}_{1\cdots n}(\lambda_0) = C_{1\cdots n}(\lambda_0)$.

Lemma 19 (Crossing symmetry). *The R-matrix has the crossing symmetry relation:*

$$(\gamma \otimes I) R_{12}(\lambda_1 - \eta, \lambda_2) (\gamma \otimes I) = b_{21}^{-1} R_{21}^{t_1}(\lambda_2, \lambda_1) \quad (5.8)$$

where $b_{21} = b(\lambda_2 - \lambda_1)$ and γ is given by

$$\gamma = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (5.9)$$

Here the crossing symmetry is slightly different from [6].

Lemma 20. (*Crossing symmetry of the monodromy matrix*)

$$\gamma_0 R_{0,1\dots n}(\lambda_0 - \eta; \xi_1, \dots, \xi_n) \gamma_0 = \left(\prod_{i=1}^n b^{-1}(\xi_i - \lambda_0) \right) R_{1\dots n, 0}^{t_0}(\lambda_0; \xi_1, \dots, \xi_n) \quad (5.10)$$

Let us introduce \dagger operation. We shall use it when we pseudo-diagonalize the B operators.

Definition 21. For $X_{1\dots n}(\lambda_1, \dots, \lambda_n) \in \text{End}(V(\lambda_1) \otimes \dots \otimes V(\lambda_n))$ we define $X_{1\dots n}^\dagger$ by

$$X_{1\dots n}^\dagger(\lambda_1, \dots, \lambda_n) = X_{1\dots n}^{t_1\dots t_n}(-\lambda_1, \dots, -\lambda_n) \quad (5.11)$$

Here we note $(X^\dagger)^\dagger = X$, and $(XY)^\dagger = Y^\dagger X^\dagger$. It is easy to show $R_{12}^\dagger = R_{21}$.

Lemma 22. Under the \dagger operation the monodromy matrix is given by the following:

$$R_{0,1\dots n}^\dagger = R_{0,1\dots n}^{-1} = R_{1\dots n, 0} \quad (5.12)$$

We define operators $A_{1\dots n}^\dagger$, $B_{1\dots n}^\dagger$, $C_{1\dots n}^\dagger$ and $D_{1\dots n}^\dagger$ by

$$R_{0,1\dots n}^\dagger = \begin{pmatrix} A_{1\dots n}^\dagger(\lambda_0) & C_{1\dots n}^\dagger(\lambda_0) \\ B_{1\dots n}^\dagger(\lambda_0) & D_{1\dots n}^\dagger(\lambda_0) \end{pmatrix}_{[0]} \quad (5.13)$$

Proposition 23. Under the \dagger operation the monodromy matrix is given by

$$\begin{aligned} R_{0,12\dots n}^\dagger &= \begin{pmatrix} A_{1\dots n}^\dagger(\lambda_0) & C_{1\dots n}^\dagger(\lambda_0) \\ B_{1\dots n}^\dagger(\lambda_0) & D_{1\dots n}^\dagger(\lambda_0) \end{pmatrix}_{[0]} \\ &= \left(\prod_{i=1}^n b(\xi_i - \lambda_0) \right) \begin{pmatrix} A_{1\dots n}(\lambda_0 - \eta) & -B_{1\dots n}(\lambda_0 - \eta) \\ -C_{1\dots n}(\lambda_0 - \eta) & D_{1\dots n}(\lambda_0 - \eta) \end{pmatrix}_{[0]} \end{aligned} \quad (5.14)$$

5.1.3 The diagonalized forms of operators A and D

Let us give the diagonalized forms of the A and D operators [6].

The following criterion for the F -basis to be non-singular should be useful.

Proposition 24. The determinants of the partial and total F matrices are given by

$$\det F_{0,1\dots n} = \prod_{j=1}^n b(\lambda_0 - \xi_j), \quad \det F_{1\dots n} = \prod_{1 \leq i < j \leq n} b(\xi_i - \xi_j) \quad (5.15)$$

Proposition 24 follows from lemma D.1 of Appendix D.

We can show the diagonalized forms of operators A and D as follows [6].

Proposition 25 (Diagonalization of A and D).

$$F_{1\dots n} D_{1\dots n}(\lambda_0) F_{1\dots n}^{-1} = \bigotimes_{i=1}^n \begin{pmatrix} b_{0i} & 0 \\ 0 & 1 \end{pmatrix}_{[i]} \quad (5.16)$$

$$\bar{F}_{1\dots n} A_{1\dots n}(\lambda_0) \bar{F}_{1\dots n}^{-1} = \bigotimes_{i=1}^n \begin{pmatrix} 1 & 0 \\ 0 & b_{0i} \end{pmatrix}_{[i]}, \quad (5.17)$$

where $b_{0i} = b(\lambda_0 - \xi_i)$

Proposition 26 (Diagonalization of A^\dagger and D^\dagger).

$$F_{1\dots n} A_{1\dots n}^\dagger(\lambda_0) F_{1\dots n}^{-1} = \bigotimes_{i=1}^n \begin{pmatrix} 1 & 0 \\ 0 & b_{i0} \end{pmatrix}_{[i]} \quad (5.18)$$

$$\bar{F}_{1\dots n} D_{1\dots n}^\dagger(\lambda_0) \bar{F}_{1\dots n}^{-1} = \bigotimes_{i=1}^n \begin{pmatrix} b_{i0} & 0 \\ 0 & 1 \end{pmatrix}_{[i]} \quad (5.19)$$

where $b_{i0} = b(\xi_i - \lambda_0)$

The derivation of the diagonalized forms and some useful formulas are briefly reviewed in Appendix D.

For a given operator $X_{1\dots n}(\lambda_1, \dots, \lambda_n) \in End(V(\lambda_1) \otimes \dots \otimes V(\lambda_n))$ we denote FAF^{-1} by \tilde{F} :

$$\tilde{X}_{1\dots n} = F_{12\dots n} X_{12\dots n} F_{12\dots n}^{-1}. \quad (5.20)$$

For instance we have $\tilde{D}_{1\dots n}(\lambda_0) = F_{1\dots n} D_{1\dots n}(\lambda_0) F_{1\dots n}^{-1}$.

5.2 Pseudo-diagonalization of the B operator

Let us recall that the matrix elements of the monodromy matrix $R_{0,1\dots L}^+$ are related to the symmetric ones as follows.

$$\begin{aligned} R_{0,12\dots L}^+(u; \xi_1, \dots, \xi_L) &= \begin{pmatrix} A_{12\dots L}^+(u; \xi_1, \dots, \xi_L) & B_{12\dots L}^+(u; \xi_1, \dots, \xi_L) \\ C_{12\dots L}^+(u; \xi_1, \dots, \xi_L) & D_{12\dots L}^+(u; \xi_1, \dots, \xi_L) \end{pmatrix}_{[0]} \\ &= \begin{pmatrix} \chi_{12\dots L} A_{12\dots L}(u; \{\xi_j\}) (\chi_{12\dots L})^{-1} & e^{-\lambda_0} \chi_{12\dots L} B_{12\dots L}(u; \{\xi_j\}) (\chi_{12\dots L})^{-1} \\ e^{\lambda_0} \chi_{12\dots L} C_{12\dots L}(u; \{\xi_j\}) (\chi_{12\dots L})^{-1} & \chi_{12\dots L} D_{12\dots L}(u; \{\xi_j\}) (\chi_{12\dots L})^{-1} \end{pmatrix}_{[0]} \end{aligned} \quad (5.21)$$

Then, from the quantum-group invariance (3.10) we have the following commutation relations:

$$B_{1\dots n}^+(\lambda) = D_{1\dots n}^+(\lambda) \Delta^{(n-1)}(X^-) - q \Delta^{(n-1)}(X^-) D_{1\dots n}^+(\lambda) \quad (5.22)$$

$$C_{1\dots n}^+(\lambda) = \Delta^{(n-1)}(X^+) D_{1\dots n}^+(\lambda) - q^{-1} D_{1\dots n}^+(\lambda) \Delta^{(n-1)}(X^+) \quad (5.23)$$

Here X^\pm are generators of $U_q(sl_2)$, and $\Delta^{(n-1)}(X^-)$ denote the tensor-product representation of $\Delta^{(n-1)}(X^-)$ acting on the n sites from the 1st to n th. We remark that more generally, we have commutation relations (3.39) for the affine quantum group $U_q(\widehat{sl}_2)$.

In this subsection we abbreviate the superscript + for the asymmetric monodromy matrix, for simplicity. In fact, the essential parts of formulas such as the fundamental commutation relations are invariant under gauge transformations if we express them in terms of the generators of the quantum affine algebra $U_q(\widehat{sl}_2)$ in the evaluation representation (3.28). Here we remark that the matrix representation of the evaluation representation of $U_q(\widehat{sl}_2)$ can be changed through gauge transformations.

Let us now introduce some symbols.

Definition 27. We define operators $\widehat{\delta}_{jk}(\lambda_j, \lambda_k)$ for j, k satisfying $0 \leq j < k \leq L$ by

$$\widehat{\delta}_{jk}(\lambda_j, \lambda_k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{kj}^{-1} & 0 & 0 \\ 1 & 0 & b_{jk}^{-1} & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}_{[jk]}, \quad (5.24)$$

where $b_{jk} = b(\lambda_j - \lambda_k)$ and $b_{kj} = b(\lambda_k - \lambda_j)$. We define $\widehat{\delta}_{1\dots n}$ and $\widehat{\delta}_{0,1\dots n}$ by

$$\begin{aligned} \widehat{\delta}_{1\dots n} &= \prod_{1 \leq j < k \leq n} \widehat{\delta}_{jk}(\lambda_j, \lambda_k) \\ \widehat{\delta}_{0,1\dots n} &= \widehat{\delta}_{01\dots n} \widehat{\delta}_{1\dots n}^{-1} = \prod_{j=1}^n \widehat{\delta}_{0j}(\lambda_0, \lambda_j) \end{aligned} \quad (5.25)$$

We define $\widehat{\delta}_i^{1\dots n}$ by

$$\widehat{\delta}_i^{1\dots n} = \widehat{\delta}_{i,i+1\dots n1\dots i-1} = \prod_{j=1; j \neq i}^n \widehat{\delta}_{ij} \quad (5.26)$$

Some useful formulas are given in Appendix D.

Let us denote $I^{\otimes m} \otimes \Delta^{(\ell-1)}(x) \otimes I^{\otimes(n-\ell-m)}$ by $\Delta_{m+1 m+2 \dots m+\ell}^{(\ell-1)}(x)$ or $\Delta_{m+1 m+2 \dots m+\ell}(x)$ for $x \in U_q(sl_2)$ in the tensor-product representation.

Lemma 28. Let X^- denote the generator of the quantum group $U_q(sl_2)$ and X_j^- the spin-1/2 representation of X^- acting on the j th site in the tensor product representation $(V^{(1)})^{\otimes n}$. We have

$$\widetilde{\Delta}_{1\dots n}(X^-) = \left(X_1^- + e_1^{11} \widetilde{\Delta}_{2\dots n}(X^-) \widetilde{A}_{2\dots n}^\dagger(\xi_1) + e_1^{22} \widetilde{D}_{2\dots n}(\xi_1) \widetilde{\Delta}_{2\dots n}(X^-) \right) \widehat{\delta}_{1,2\dots n} \quad (5.27)$$

Proof. Making use of (D.12) we show $F_{1\dots n}^{-1} = F_{n\dots 2,1}^{t_1\dots t_n} F_{2\dots n}^{-1} \widehat{\delta}_{1,2\dots n}$. We have

$$\widetilde{\Delta}_{1\dots n}(X^-) = F_{2\dots n} F_{1,2\dots n} \Delta^{(n-1)}(X^-) F_{n\dots 2,1}^{t_1\dots t_n} F_{2\dots n}^{-1} \widehat{\delta}_{1,2\dots n} \quad (5.28)$$

Putting $F_{1,2\dots n} = e_1^{11} + e_1^{22} R_{1,2\dots n}$ and $F_{n\dots 2,1}^{t_1\dots t_n} = e_1^{22} + R_{2\dots n,1} e_1^{11}$, we have

$$\begin{aligned} F_{1,2\dots n} \Delta^{(n-1)}(X^-) F_{n\dots 2,1}^{t_1\dots t_n} &= (e_1^{11} + e_1^{22} R_{1,2\dots n}) \Delta^{(n-1)}(X^-) (e_1^{22} + R_{2\dots n,1} e_1^{11}) \\ &= e_1^{11} \Delta^{(n-1)}(X^-) e_1^{22} + e_1^{11} \Delta^{(n-1)}(X^-) R_{2\dots n,1} e_1^{11} \\ &\quad + e_1^{22} R_{1,2\dots n} \Delta^{(n-1)}(X^-) e_1^{22} + e_1^{22} R_{1,2\dots n} \Delta^{(n-1)}(X^-) R_{2\dots n,1} e_1^{11} \\ &= 0 + e_1^{11} \Delta_{2\dots n}(X^-) R_{2\dots n,1} e_1^{11} + e_1^{22} R_{1,2\dots n} \Delta_{2\dots n}(X^-) e_1^{22} + X_1^- . \end{aligned} \quad (5.29)$$

Here we have made use of the following:

$$\Delta^{(n-1)}(X^-) = \Delta^{(n-2)} \Delta(X^-) = X^- \otimes \Delta^{(n-2)}(K^{-1}) + I \otimes \Delta^{(n-2)}(X^-) . \quad (5.30)$$

We thus have

$$\widetilde{\Delta}_{1\dots n}(X^-) = \left(X_1^- + e_1^{11} \widetilde{\Delta}_{2\dots n}(X^-) \widetilde{R}_{2\dots n,1} e_1^{11} + e_1^{22} \widetilde{R}_{1,2\dots n} \widetilde{\Delta}_{2\dots n}(X^-) e_1^{11} \right) \widehat{\delta}_{1,2\dots n} . \quad (5.31)$$

We obtain the case of n from (5.12). \square

Lemma 29. In the tensor-product representation $(V^{(1)})^{\otimes n}$ we have

$$\tilde{\Delta}_{1\cdots n}(X^-) = \sum_{i=1}^n X_i^- \hat{\delta}_i^{1\cdots n}. \quad (5.32)$$

Proof. We show it by induction on n . The case of $n = 1$ is trivial. Let us assume the case of $n - 1$. In eq. (5.27), the first term gives the following: $X_1^- \hat{\delta}_{1,2\cdots n} = X_1^- \hat{\delta}_1^{1\cdots n}$. Assuming (5.32) for $\tilde{\Delta}_{2\cdots n}$ and putting it into the second term of (5.27), we have

$$\begin{aligned} e_1^{11} \tilde{\Delta}_{2\cdots n}(X^-) \tilde{A}^\dagger_{2\cdots n}(\xi_1) \hat{\delta}_{1,2\cdots n} &= e_1^{11} \sum_{i=2}^n X_i^- \hat{\delta}_i^{2\cdots n} \bigotimes_{k=2}^n \begin{pmatrix} 1 & 0 \\ 0 & b_{k1} \end{pmatrix}_{[k]} \hat{\delta}_{1,2\cdots n} \\ &= \sum_{i=2}^n X_i^- \hat{\delta}_i^{1\cdots n} e_1^{11} \\ &= e_1^{11} \sum_{i=2}^n X_i^- \hat{\delta}_i^{1\cdots n} \end{aligned} \quad (5.33)$$

Here we have made use of (5.18). Similarly, we have

$$e_1^{22} \tilde{D}_{2\cdots n}(\xi_1) \tilde{\Delta}_{2\cdots n}(X^-) e_1^{22} \hat{\delta}_{1,2\cdots n} = e_1^{22} \sum_{i=2}^n X_i^- \hat{\delta}_i^{1\cdots n} \quad (5.34)$$

Thus, we have the case of n as follows.

$$\begin{aligned} \tilde{\Delta}_{1\cdots n}(X^-) &= X_1^- \hat{\delta}_1^{1\cdots n} + (e_1^{11} + e_1^{22}) \sum_{i=2}^n X_i^- \hat{\delta}_i^{1\cdots n} \\ &= \sum_{i=1}^n X_i^- \hat{\delta}_i^{1\cdots n}. \end{aligned} \quad (5.35)$$

□

From the fundamental commutation relation (5.22) we have the following:

Lemma 30. In the tensor product $(V^{(1)})^{\otimes n}$ we have

$$\tilde{B}_{1\cdots n}(\lambda) = \sum_{i=1}^n X_i^- (\tilde{D}_{1\cdots i-1,i+1\cdots n}(\lambda) - q \tilde{D}_{1\cdots n}(\lambda)) \hat{\delta}_i^{1\cdots n}. \quad (5.36)$$

Proof. We transform the both sides of the fundamental commutation relation (5.22) by $F_{1\cdots n}$, and put (5.16) and (5.32) into it, we have the following:

$$\begin{aligned} \tilde{B}_{1\cdots n}(\lambda) &= \tilde{D}_{1\cdots n}(\lambda) \tilde{\Delta}_{1\cdots n}(X^-) - q \tilde{\Delta}_{1\cdots n}(X^-) \tilde{D}_{1\cdots n}(\lambda) \\ &= \bigotimes_{j=1}^n \begin{pmatrix} b_{0j} & 0 \\ 0 & 1 \end{pmatrix}_{[j]} \sum_{i=1}^n X_i^- \hat{\delta}_i^{1\cdots n} - q \sum_{i=1}^n X_i^- \hat{\delta}_i^{1\cdots n} \bigotimes_{j=1}^n \begin{pmatrix} b_{0j} & 0 \\ 0 & 1 \end{pmatrix}_{[j]} \\ &= \sum_{i=1}^n X_i^- (\tilde{D}_{1\cdots i-1,i+1\cdots n}(\lambda) - q \tilde{D}_{1\cdots n}(\lambda)) \hat{\delta}_i^{1\cdots n}. \end{aligned} \quad (5.37)$$

□

Proposition 31 (Pseudo-diagonalization of B operator). *We have*

$$\tilde{B}_{1\dots n}(\lambda) = \sum_{i=1}^n c_{0i}^- X_i^- \bigotimes_{j=1; j \neq i}^n \begin{pmatrix} b_{0j} & 0 \\ 0 & b_{ji}^{-1} \end{pmatrix}_{[j]}, \quad (5.38)$$

where $b_{0i} = b(\lambda_0 - \xi_i)$, $b_{ji} = b(\xi_j - \xi_i)$ and $c_{0i}^- = c^-(\lambda_0 - \xi_i) = \exp(-(\lambda_0 - \xi_i))c(\lambda_0 - \xi_i)$.

Proof. Let us denote $b_{0i} = \sinh(\lambda - \xi_i)/\sinh(\lambda - \xi_i + \eta)$ and $c_{0i} = \sinh(\eta)/\sinh(\lambda - \xi_i + \eta)$, by b_{0i} and c_{0i} respectively, . Putting $1 - qb_{01} = c_{0i}^-$ in (5.36) we show

$$\tilde{B}_{1\dots n}(\lambda) = \sum_{i=1}^n c_{0i}^- X_i^- \tilde{D}_{1\dots i-1 i+1\dots n} \hat{\delta}_i^{1\dots n}. \quad (5.39)$$

After some calculation, we have (5.38). \square

Similarly, making use of lemmas E.1, E.2 and E.3, we can show the diagonalized form of operator C .

Proposition 32. *Let X_i^+ denote the spin-1/2 representation of X^+ acting on the i th site in the tensor product representation. We have*

$$\tilde{C}_{1\dots n}(\lambda_0) = \sum_{i=1}^n c_{0i}^+ X_i^+ \bigotimes_{j=1; j \neq i}^n \begin{pmatrix} b_{0j} b_{ij}^{-1} & 0 \\ 0 & 1 \end{pmatrix}_{[j]}, \quad (5.40)$$

where $b_{0i} = b(\lambda_0 - \xi_i)$, $b_{ji} = b(\xi_j - \xi_i)$ and $c_{0i}^+ = c^+(\lambda_0 - \xi_i)$.

5.3 Pseudo-diagonalized forms of the symmetric B and C operators

Let us show the pseudo-diagonalized forms of the B and C operators of the symmetric monodromy matrix $R_{0,1\dots n}$. Here we recall that expressions (5.38) and (5.40) are for $\tilde{B}_{12\dots n}^+(\lambda)$ and $\tilde{C}_{12\dots n}^+(\lambda)$, respectively. They are matrix elements of the asymmetric monodromy matrix $R_{0,1\dots n}^+ = \chi_{01\dots n} R_{0,1\dots n} \chi_{01\dots n}^{-1}$. We have the following relations:

$$\begin{aligned} B_{12\dots n}^+(\lambda) &= e^{-\lambda} \chi_{01\dots n} R_{0,1\dots n} \chi_{01\dots n}^{-1} \\ C_{12\dots n}^+(\lambda) &= e^\lambda \chi_{01\dots n} R_{0,1\dots n} \chi_{01\dots n}^{-1} \end{aligned} \quad (5.41)$$

Therefore, applying the inverse gauge transformation $\bar{\chi}$ to (5.38) and (5.40), we obtain

$$\tilde{B}_{1\dots n}(\lambda) = \sum_{i=1}^n c_{0i} \sigma_i^- \bigotimes_{j=1; j \neq i}^n \begin{pmatrix} b_{0j} & 0 \\ 0 & b_{ji}^{-1} \end{pmatrix}_{[j]}, \quad (5.42)$$

and

$$\tilde{C}_{1\dots n}(\lambda_0) = \sum_{i=1}^n c_{0i} \sigma_i^+ \bigotimes_{j=1; j \neq i}^n \begin{pmatrix} b_{0j} b_{ij}^{-1} & 0 \\ 0 & 1 \end{pmatrix}_{[j]}. \quad (5.43)$$

Here we recall $c_{0i} = \sinh(\eta)/\sinh(\lambda - \xi_i + \eta)$.

We should remark that expressions (5.42) and (5.43) coincide with eq. (2.29) and (2.30) of Ref. [7], respectively.

6 Scalar products formulas

6.1 Formula for higher-spin scalar products

Let us consider the case of tensor product of spin- s representations. We recall that $\ell = 2s$ and $L = \ell N_s$. We introduce parameters $\xi_j^{(\ell;\epsilon)}$ for $j = 1, 2, \dots, L$, as follows:

$$\xi_{(k-1)\ell+j}^{(\ell;\epsilon)} = \zeta_k - (j-1)\eta + \ell\eta/2 + \epsilon r_j \quad j = 1, \dots, \ell; k = 1, \dots, N_s. \quad (6.1)$$

Here r_j ($j = 1, 2, \dots, \ell$) are distinct and nonzero parameters, and ϵ is an arbitrary small number. We also introduce the following symbol:

$$P_{1\dots L}^{(\ell)} = \prod_{j=1}^{N_s} P_{(j-1)\ell+1}^{(\ell)}, \quad P_{1\dots L}^{(\ell)\bar{\chi}} = \prod_{j=1}^{N_s} \left(P_{(j-1)\ell+1}^{(\ell)} \right)^{\bar{\chi}} \quad (6.2)$$

Here we recall that B operator acting on the tensor product of spin- s representations, $(V^{(2s)})^{\otimes N_s}$, is given by B operator acting on the tensor product of spin-1/2 representations $(V^{(1)})^{\otimes L}$ with $L = N_s\ell$ and multiplied by the projection operators:

$$B_{1\dots N_s}^{(\ell)}(u; \zeta_1, \dots, \zeta_{N_s}) = P_{1\dots L}^{(\ell)} B_{1\dots L}^{(1)}(u; \xi_1^{(\ell)}, \dots, \xi_L^{(\ell)}) P_{1\dots L}^{(\ell)}.$$

We now define the scalar product for the spin- $\ell/2$ case as follows.

Definition 33. Let $\{\lambda_\alpha\}$ ($\alpha = 1, 2, \dots, n$) be a set of solutions of the Bethe ansatz equations and $\{\mu_j\}$ ($j = 1, 2, \dots, n$) be arbitrary numbers. We define the scalar product $S_n^{(\ell)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\zeta_k\})$ by the following:

$$S_n^{(\ell)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\zeta_k\}) = \langle 0 | C^{(\ell)}(\mu_1) \cdots C^{(\ell)}(\mu_n) B^{(\ell)}(\lambda_1) \cdots B^{(\ell)}(\lambda_n) | 0 \rangle \quad (6.3)$$

Here $C^{(\ell)}(\mu_j)$ and $B^{(\ell)}(\lambda_\alpha)$ abbreviate $C_{1\dots N_s}^{(\ell)}(\mu_j; \{\zeta_j\})$ and $B_{1\dots N_s}^{(\ell)}(\lambda_\alpha; \{\zeta_j\})$, respectively, and ζ_k denote the centers of ℓ_k -strings of the inhomogeneous parameters $\{\xi_k^{(\ell)}\}$.

We calculate the scalar product for the higher-spin XXZ chains by the formula in the next proposition.

Proposition 34. Let $\{\lambda_\alpha\}$ satisfy the Bethe ansatz equations for the spin- $\ell/2$ case. The scalar product of the spin- $\ell/2$ XXZ spin chain is reduced into that of the spin-1/2 XXZ spin chain as follows:

$$S_n^{(\ell)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\zeta_k\}) = \lim_{\epsilon \rightarrow 0} \left[S_n^{(1)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k^{(\ell;\epsilon)}\}) \right] \quad (6.4)$$

Proof. We now calculate the scalar product making use of eq. (4.7) of lemma 14 as follows.

$$\begin{aligned} & \langle 0 | C_{1\dots N_s}^{(\ell)}(\mu_1; \{\zeta_j\}) \cdots C_{1\dots N_s}^{(\ell)}(\mu_n; \{\zeta_j\}) B_{1\dots N_s}^{(\ell)}(\lambda_1; \{\zeta_j\}) \cdots B_{1\dots N_s}^{(\ell)}(\lambda_n; \{\zeta_j\}) | 0 \rangle \\ &= \langle 0 | \left(P_{1\dots L}^{(\ell)\bar{\chi}} C_{1\dots L}^{(1)}(\mu_1; \{\xi_j^{(\ell)}\}) P_{1\dots L}^{(\ell)\bar{\chi}} \right) \cdots \left(P_{1\dots L}^{(\ell)\bar{\chi}} C_{1\dots L}^{(1)}(\mu_n; \{\xi_j^{(\ell)}\}) P_{1\dots L}^{(\ell)\bar{\chi}} \right) \\ & \quad \cdot \left(P_{1\dots L}^{(\ell)\bar{\chi}} B_{1\dots L}^{(1)}(\lambda_1; \{\xi_j^{(\ell)}\}) P_{1\dots L}^{(\ell)\bar{\chi}} \right) \cdots \left(P_{1\dots L}^{(\ell)\bar{\chi}} B_{1\dots L}^{(1)}(\lambda_n; \{\xi_j^{(\ell)}\}) P_{1\dots L}^{(\ell)\bar{\chi}} \right) | 0 \rangle \\ &= \langle 0 | P_{1\dots L}^{(\ell)\bar{\chi}} \times C_{1\dots L}^{(1)}(\mu_1; \{\xi_j^{(\ell)}\}) \cdots C_{1\dots L}^{(1)}(\mu_n; \{\xi_j^{(\ell)}\}) \\ & \quad \cdot B_{1\dots L}^{(1)}(\lambda_1; \{\xi_j^{(\ell)}\}) \cdots B_{1\dots L}^{(1)}(\lambda_n; \{\xi_j^{(\ell)}\}) \times P_{1\dots L}^{(\ell)\bar{\chi}} | 0 \rangle \end{aligned} \quad (6.5)$$

Here we note that we have $\langle 0|P_{1\dots L}^{(\ell)} = \langle 0|$ and $P_{1\dots L}^{(\ell)}|0\rangle = |0\rangle$. Moreover, we have $\langle 0|P_{1\dots L}^{(\ell)\bar{\chi}} = \langle 0|$ and $P_{1\dots L}^{(\ell)\bar{\chi}}|0\rangle = |0\rangle$. We thus have

$$\begin{aligned} & \langle 0|C_{1\dots N_s}^{(\ell)}(\mu_1; \{\zeta_j\}) \cdots C_{1\dots N_s}^{(\ell)}(\mu_n; \{\zeta_j\}) B_{1\dots N_s}^{(\ell)}(\lambda_1; \{\zeta_j\}) \cdots B_{1\dots N_s}^{(\ell)}(\lambda_n; \{\zeta_j\})|0\rangle \\ = & \langle 0|C_{1\dots L}^{(1)}(\mu_1; \{\xi_j^{(\ell)}\}) \cdots C_{1\dots L}^{(1)}(\mu_n; \{\xi_j^{(\ell)}\}) \cdot B_{1\dots L}^{(1)}(\lambda_1; \{\xi_j^{(\ell)}\}) \cdots B_{1\dots L}^{(1)}(\lambda_n; \{\xi_j^{(\ell)}\})|0\rangle \end{aligned} \quad (6.6)$$

We evaluate the last line through the following limit of sending ϵ to zero:

$$\begin{aligned} & \langle 0|C_{1\dots L}^{(1)}(\mu_1; \{\xi_j^{(\ell)}\}) \cdots C_{1\dots L}^{(1)}(\mu_n; \{\xi_j^{(\ell)}\}) \cdot B_{1\dots L}^{(1)}(\lambda_1; \{\xi_j^{(\ell)}\}) \cdots B_{1\dots L}^{(1)}(\lambda_n; \{\xi_j^{(\ell)}\})|0\rangle \\ = & \lim_{\epsilon \rightarrow 0} \langle 0|C_{1\dots L}^{(1)}(\mu_1; \{\xi_j^{(\ell;\epsilon)}\}) \cdots C_{1\dots L}^{(1)}(\mu_n; \{\xi_j^{(\ell;\epsilon)}\}) \\ & \quad \times B_{1\dots L}^{(1)}(\lambda_1; \{\xi_j^{(\ell;\epsilon)}\}) \cdots B_{1\dots L}^{(1)}(\lambda_n; \{\xi_j^{(\ell;\epsilon)}\})|0\rangle \\ = & \lim_{\epsilon \rightarrow 0} \left[\langle 0|\tilde{C}_{1\dots L}^{(1)}(\mu_1; \{\xi_j^{(\ell;\epsilon)}\}) \cdots \tilde{C}_{1\dots L}^{(1)}(\mu_n; \{\xi_j^{(\ell;\epsilon)}\}) \right. \\ & \quad \left. \times \tilde{B}_{1\dots L}^{(1)}(\lambda_1; \{\xi_j^{(\ell;\epsilon)}\}) \cdots \tilde{B}_{1\dots L}^{(1)}(\lambda_n; \{\xi_j^{(\ell;\epsilon)}\}) \right] |0\rangle \end{aligned} \quad (6.7)$$

□

We evaluate the spin-1/2 scalar product taking the limit of sending ϵ to 0, so that we can make the determinant of $F_{L\dots 21}$ being nonzero. Here we remark that the operator $F_{L\dots 21}$ appears in the pseudo-diagonalization process of the B and C operators, as shown in Section 5, and also that the determinant of $F_{L\dots 21}$ vanishes at $\epsilon = 0$, when parameters ξ_j are given by eq. (6.1). In fact, if we put some inhomogeneous parameters ξ_j in the form of a “complete ℓ -string” [29], that is, for some integers ℓ, m and a constant z , we have $\xi_{m+j} = z - j\eta$ for $j = 1, 2, \dots, \ell$, then the determinant of $F_{L\dots 21}$ vanishes. Here we also note that $\det F_{12\dots L} \neq 0$ even at $\epsilon = 0$.

Let us discuss the mixed spin case. We set the inhomogeneous parameters as follows:

$$\xi_{\ell_1+\dots+\ell_{k-1}+j}^{(\ell;\epsilon)} = \zeta_k - (j-1)\eta + (\ell-1)\eta/2 + \epsilon r_j \quad j = 1, \dots, \ell; k = 1, \dots, r. \quad (6.8)$$

Let us define $P_{12\dots L}^{(\ell)}$ and $P_{12\dots L}^{(\ell)\bar{\chi}}$ by

$$\begin{aligned} P_{12\dots L}^{(\ell)} &= \prod_{k=1}^r P_{\ell(k-1)+1}^{(\ell_k)} \\ P_{12\dots L}^{(\ell)\bar{\chi}} &= \prod_{k=1}^r \left(P_{\ell(k-1)+1}^{(\ell_k)} \right)^{\bar{\chi}} \end{aligned} \quad (6.9)$$

It is easy to see the following:

$$\begin{aligned} \langle 0|P_{12\dots L}^{(\ell)} &= \langle 0|, \quad P_{12\dots L}^{(\ell)}|0\rangle = |0\rangle, \\ \langle 0|P_{12\dots L}^{(\ell)\bar{\chi}} &= \langle 0|, \quad P_{12\dots L}^{(\ell)\bar{\chi}}|0\rangle = |0\rangle \end{aligned} \quad (6.10)$$

We now define the scalar product for the mixed spin case as follows.

$$S_n^{(\ell)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\zeta_k\}) = \langle 0|C^{(\ell)}(\mu_1) \cdots C^{(\ell)}(\mu_n) B^{(\ell)}(\lambda_1) \cdots B^{(\ell)}(\lambda_n)|0\rangle \quad (6.11)$$

Here $C^{(\ell)}(\mu_j)$ and $B^{(\ell)}(\lambda_\alpha)$ abbreviate $C_{1\dots r}^{(\ell)}(\mu_j; \{\zeta_j\})$ and $B_{1\dots r}^{(\ell)}(\lambda_\alpha; \{\zeta_j\})$, respectively.

Proposition 35. Let $\{\lambda_\alpha\}$ satisfy the Bethe ansatz equations for the mixed spin case. The scalar product of the mixed-spin XXZ spin chain is reduced into that of the spin-1/2 XXZ spin chain as follows:

$$S_n^{(\ell)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\zeta_k\}) = \lim_{\epsilon \rightarrow 0} \left[S_n^{(1)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k^{(\ell, \epsilon)}\}) \right] \quad (6.12)$$

6.2 Determinant expressions of the scalar products

Let us review the result of the spin-1/2 case [7]. Suppose that λ_α for $\alpha = 1, 2, \dots, n$, are solutions of the Bethe ansatz equations with in homogeneous parameters ξ_j for $j = 1, 2, \dots, L$, the scalar product is defined by

$$S_n(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k\}) = \langle 0 | \prod_{j=1}^n C(\mu_j; \{\xi_k\}) \prod_{\alpha=1}^n B(\lambda_\alpha; \{\xi_k\}) | 0 \rangle \quad (6.13)$$

Here μ_j for $j = 1, 2, \dots, n$ are arbitrary. We note that $S_n(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k\})$ has been denoted by $S_n^{(1)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k\})$ in the last subsection. Then, the exact expression of the scalar product has been shown through the pseudo-diagonalized forms of the B and C operators as follows [7]:

$$\begin{aligned} S_n(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k\}) &= \langle 0 | \prod_{j=1}^n \tilde{C}(\mu_j) \prod_{\alpha=1}^n \tilde{B}(\lambda_\alpha) | 0 \rangle \\ &= \frac{\prod_{\alpha=1}^n \prod_{j=1}^n \sinh(\mu_j - \lambda_\alpha)}{\prod_{j>k} \sinh(\mu_k - \mu_j) \prod_{\alpha<\beta} \sinh(\lambda_\beta - \lambda_\alpha)} \det T(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k\}) \end{aligned} \quad (6.14)$$

Here the matrix elements T_{ab} for $a, b = 1, \dots, n$, are given by

$$T_{ab} = \frac{\partial}{\partial \lambda_\alpha} \tau(\mu_b, \{\lambda_k\}; \{\xi_k\}) \quad (6.15)$$

where

$$\tau(\mu, \{\lambda_k\}; \{\xi_k\}) = a(\mu) \prod_{k=1}^n b^{-1}(\lambda_k - \mu) + d(\mu; \{\xi_j\}; \{\xi_k\}) \prod_{k=1}^n b^{-1}(\mu - \lambda_k) \quad (6.16)$$

and

$$a(\mu) = 1, \quad d(\mu; \{\xi_j\}) = \prod_{j=1}^L b(\mu - \xi_j). \quad (6.17)$$

Let us express the scalar product of the higher-spin case in terms of the determinant of the matrix T . In the tensor product of spin- $\ell/2$ representations we have

$$\begin{aligned} S_n^{(\ell)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\zeta_k\}) &= \frac{\prod_{\alpha=1}^n \prod_{j=1}^n \sinh(\mu_j - \lambda_\alpha)}{\prod_{j>k} \sinh(\mu_k - \mu_j) \prod_{\alpha<\beta} \sinh(\lambda_\beta - \lambda_\alpha)} \det T(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k^{(\ell)}\}) \end{aligned} \quad (6.18)$$

In the mixed-spin case we have

$$\begin{aligned} S_n^{(\ell)}(\{\mu_j\}, \{\lambda_\alpha\}; \{\zeta_k\}) &= \frac{\prod_{\alpha=1}^n \prod_{j=1}^n \sinh(\mu_j - \lambda_\alpha)}{\prod_{j>k} \sinh(\mu_k - \mu_j) \prod_{\alpha<\beta} \sinh(\lambda_\beta - \lambda_\alpha)} \det T(\{\mu_j\}, \{\lambda_\alpha\}; \{\xi_k^{(\ell)}\}) \end{aligned} \quad (6.19)$$

6.3 Norms of the Bethe states in the higher-spin case

For two sets of n parameters, μ_1, \dots, μ_n and $\lambda_1, \dots, \lambda_n$, we define matrix elements H_{ab} by

$$H_{ab}(\{\mu_j\}, \{\lambda_\alpha; \{\xi_k\}\}) = \frac{\sinh \eta}{\sinh(\lambda_a - \mu_b)} \left(\frac{a(\mu_b)}{d(\mu_b)} \prod_{k=1; k \neq a}^n \sinh(\lambda_k - \mu_b + \eta) - \prod_{k=1; k \neq a}^n \sinh(\lambda_k - \mu_b - \eta) \right) \quad (6.20)$$

Let us assume that $\lambda_1, \dots, \lambda_n$ are solutions of the Bethe ansatz equations. We have

$$\det H(\{\lambda_\alpha\}, \{\mu_j\}; \{\xi_k\}) = \det T(\{\lambda_\alpha\}, \{\mu_j\}; \{\xi_k\}) \prod_{\alpha=1}^n \prod_{j=1}^n \sinh(\mu_j - \lambda_\alpha) \left(\prod_{j=1}^n d(\mu_j) \right)^{-1} \quad (6.21)$$

Let us now take the limit of sending μ_j to λ_j for each j . Then we have

$$\lim_{\mu_j \rightarrow \lambda_j} \det H(\{\lambda_\alpha\}, \{\mu_j\}; \{\xi_k\}) = \sinh^n \eta \prod_{\beta=1}^n \prod_{m=1; m \neq \beta}^n \sinh(\lambda_m - \lambda_\beta - \eta) \cdot \det \Phi'(\{\lambda_\alpha\}) \quad (6.22)$$

where matrix elements Φ'_{ab} for $a, b = 1, \dots, n$, are given by

$$\Phi'_{ab}(\{\lambda_\alpha\}; \{\xi\}) = -\frac{\partial}{\partial \lambda_b} \left(\frac{a(\lambda_a; \{\xi_j\})}{d(\lambda_b; \{\xi_j\})} \prod_{k=1; k \neq a}^n \frac{b(\lambda_a - \lambda_k)}{b(\lambda_k - \lambda_a)} \right) \quad (6.23)$$

Suppose that λ_α for $n = 1, 2, \dots, n$ are solutions of the Bethe ansatz equations. Gaudin's formula for the square of the norm of the Bethe state is given by

$$\begin{aligned} N_n(\{\lambda_\alpha\}; \{\xi_j\}) &= \langle 0 | \prod_{j=1}^n C(\lambda_j) \prod_{j=1}^n B(\lambda_k) | 0 \rangle \\ &= \sinh^n \eta \prod_{\alpha, \beta=1; \alpha \neq \beta}^n b^{-1}(\lambda_\alpha - \lambda_\beta) \cdot \det \Phi'(\lambda_\alpha; \{\xi_j\}) \end{aligned} \quad (6.24)$$

Let us define the norm of the Bethe state for the mixed spin case of ℓ as follows.

$$N_n^{(\ell)}(\{\lambda_\alpha\}; \{\xi_k^{(\ell)}\}) = \langle 0 | \prod_{j=1}^n C^{(\ell)}(\lambda_j) \prod_{j=1}^n B^{(\ell)}(\lambda_k) | 0 \rangle \quad (6.25)$$

Then we have

$$N_n^{(\ell)}(\{\lambda_\alpha\}; \{\xi_k^{(\ell)}\}) = \sinh^n \eta \prod_{\alpha, \beta=1; \alpha \neq \beta}^n b^{-1}(\lambda_\alpha - \lambda_\beta) \cdot \det \Phi'(\lambda_\alpha; \{\xi_j^\ell\}) \quad (6.26)$$

7 Form factors and the inverse-scattering for the higher-spin case

7.1 Formulas of the quantum inverse scattering problem

Let us briefly review the derivation of the fundamental lemma of the quantum-inverse scattering problem for the spin-1/2 XXZ spin chain [6]. It will thus becomes clear how the pseudo-diagonalization of B and C are important.

Proposition 36. Let us denote by p_q sequence $p_q = (1, 2, \dots, n)$. Recall the notation $\tilde{R}_{0,p_q} = F_{12\dots n} R_{0,12\dots n} F_{12\dots n}^{-1}$. Then, \tilde{R}_{0,p_q} is invariant under any permutation. We have

$$\tilde{R}_{0,p_q} = \tilde{R}_{0,\sigma(p_q)}, \quad \text{for } \sigma \in \mathcal{S}_n. \quad (7.1)$$

We shall show (7.1) in Appendix B.

The following lemma plays a central role in the quantum inverse-scattering problem [7].

Lemma 37. For arbitrary inhomogeneous parameters $\xi_1, \xi_2, \dots, \xi_L$ we have

$$x_i = \prod_{\alpha=1}^{i-1} (A + D)(\xi_\alpha) \operatorname{tr}_0(x_0 R_{0,1\dots L}(\xi_i)) \prod_{\alpha=i+1}^L (A + D)(\xi_\alpha) \quad (7.2)$$

Proof. For any operator x_0 defined on the auxiliary space, we have

$$\begin{aligned} \operatorname{tr}_0(x_0 R_{0,1\dots L}(\lambda = \xi_i)) &= F_{12\dots L}^{-1} \operatorname{tr}_0(x_0 \tilde{R}_{0,1\dots L}(\lambda = \xi_i)) F_{12\dots L} \\ &= F_{12\dots L}^{-1} \operatorname{tr}_0(x_0 \tilde{R}_{i,i+1\dots L1\dots i-1}(\lambda = \xi_i)) F_{12\dots L} \\ &= F_{12\dots L}^{-1} F_{i\dots L1\dots i-1} \operatorname{tr}_0(x_0 R_{i,i+1\dots L1\dots i-1}(\lambda = \xi_i)) F_{i\dots L1\dots i-1}^{-1} F_{12\dots L} \\ &= (F_{i\dots L1\dots i-1}^{-1} F_{12\dots L})^{-1} x_i (A(\xi_i) + D(\xi_i)) (F_{i\dots L1\dots i-1}^{-1} F_{12\dots L}) \end{aligned}$$

Here we have used (7.1), i.e. $\tilde{R}_{0,1\dots L} = \tilde{R}_{i,i+1\dots L1\dots i-1}$. From the expression of $F_{i\dots L1\dots i-1}^{-1} F_{12\dots L}$ we now have

$$\operatorname{tr}_0(x_0 R_{0,1\dots L}) = \prod_{\alpha=1}^{i-1} ((A + D)(\xi_\alpha))^{-1} \cdot x_i \cdot \prod_{\alpha=1}^i (A + D)(\xi_\alpha). \quad (7.3)$$

□

7.2 Quantum inverse-scattering problem for the higher-spin operators

Let us consider monodromy matrix $T_{0,1\dots \ell N_s}^+$. Here we recall $L = \ell N_s$. For simplicity, we shall suppress the superscript ‘+’ for A, B, C and D operators through this subsection.

We recall the following: $\Delta^{(n-1)}(K) = K^{\otimes n}$ and

$$\begin{aligned} \Delta^{(n-1)}(X^+) &= \sum_{j=1}^n K^{\otimes(j-1)} \otimes X_j^+ \otimes I^{\otimes(n-j)}, \\ \Delta^{(n-1)}(X^-) &= \sum_{j=1}^n I^{\otimes(j-1)} \otimes X_j^- \otimes (K^{-1})^{\otimes(n-j)}. \end{aligned} \quad (7.4)$$

It is useful to note that for $i = 1, 2, \dots, \ell N_s$ we have

$$K_i = \prod_{\alpha=1}^{i-1} (A + D)(\xi_\alpha) (qA + q^{-1}D)(\xi_i) \prod_{\alpha=i+1}^{\ell N_s} (A + D)(\xi_\alpha). \quad (7.5)$$

In the tensor product of $2N_s$ spin-1/2 representations, $\left(V_1^{(1)} \otimes V_2^{(1)}\right)^{\otimes N_s}$, we have

$$\begin{aligned}\Delta_{12}(X^-) &= (X_1^- \otimes K_2^{-1} + I_1 \otimes X_2^-) \otimes I^{\otimes 2(N_s-1)} \\ &= \left\{ \left(B(\xi_1) \cdot \prod_{\alpha=2}^{2N_s} (A+D)(\xi_\alpha) \right) \cdot \left((A+D)(\xi_1)(q^{-1}A+qD)(\xi_2) \prod_{\alpha=3}^{2N_s} (A+D)(\xi_\alpha) \right) \right. \\ &\quad \left. + (A+D)(\xi_1) \cdot B(\xi_2) \cdot \prod_{\alpha=3}^{2N_s} (A+D)(\xi_\alpha) \right\} \end{aligned}\quad (7.6)$$

In the ℓ th tensor product of spin-1/2 representations, $V_1^{(1)} \otimes \cdots \otimes V_\ell^{(1)}$, we have

$$P_{1\cdots\ell}^{(\ell)} \cdot \Delta^{(\ell-1)}(X^\pm) \cdot P_{1\cdots\ell}^{(\ell)} = P_{1\cdots\ell}^{(\ell)} \cdot \Delta^{(\ell-1)}(X^\pm) = \Delta^{(\ell-1)}(X^\pm) \cdot P_{1\cdots\ell}^{(\ell)} \quad (7.7)$$

In the tensor product of spin- $\ell/2$ representations, $V_1^{(\ell)} \otimes \cdots \otimes V_{N_s}^{(\ell)}$, we have for $i = 1, 2, \dots, N_s$ the following relations:

$$\begin{aligned}X_i^{-(\ell+)} &= \Delta_{(i-1)\ell+1 \cdots i\ell}^{(\ell-1)}(X^-) \\ &= \sum_{k=1}^{\ell} \prod_{\alpha=1}^{(i-1)\ell+k-1} (A+D)(\xi_\alpha) \cdot B(\xi_{(i-1)\ell+k}) \cdot \prod_{\alpha=(i-1)\ell+k+1}^{\ell N_s} (A+D)(\xi_\alpha) \\ &\quad \times \prod_{j=k+1}^{\ell} \prod_{\alpha=1}^{(i-1)\ell+j-1} (A+D)(\xi_\alpha) \cdot (q^{-1}A+qD)(\xi_{(i-1)\ell+j}) \cdot \prod_{\alpha=(i-1)\ell+j+1}^{\ell N_s} (A+D)(\xi_\alpha) \\ &= \sum_{k=1}^{\ell} \prod_{\alpha=1}^{(i-1)\ell+k-1} (A+D)(\xi_\alpha) \cdot B(\xi_{(i-1)\ell+k}) \cdot \\ &\quad \times \prod_{j=k+1}^{\ell} (q^{-1}A+qD)(\xi_{(i-1)\ell+j}) \cdot \prod_{\alpha=i\ell+1}^{\ell N_s} (A+D)(\xi_\alpha). \end{aligned}\quad (7.8)$$

Here we have made use of the following:

$$\prod_{\alpha=1}^{\ell N_s} (A+D)(\xi_\alpha) = I^{\otimes \ell N_s}. \quad (7.9)$$

Similarly, we can express $X_i^{+(\ell+)}$ and $K_i^{(\ell+)}$ as follows.

$$\begin{aligned}X_i^{+(\ell+)} &= \Delta_{(i-1)\ell+1 \cdots i\ell}^{(\ell-1)}(X^+) \\ &= \sum_{k=1}^{\ell} \prod_{\alpha=1}^{(i-1)\ell} (A+D)(\xi_\alpha) \cdot \prod_{j=1}^{k-1} (qA+q^{-1}D)(\xi_{(i-1)\ell+j}) \cdot C(\xi_{(i-1)\ell+k}) \\ &\quad \prod_{\alpha=(i-1)\ell+k+1}^{\ell N_s} (A+D)(\xi_\alpha). \end{aligned}\quad (7.10)$$

$$\begin{aligned}K_i^{(\ell+)} &= \Delta_{(i-1)\ell+1 \cdots i\ell}^{(\ell-1)}(K) \\ &= \prod_{\alpha=1}^{(i-1)\ell} (A+D)(\xi_\alpha) \prod_{j=1}^{\ell} (qA+q^{-1}D)(\xi_{(i-1)\ell+j}) \prod_{\alpha=i\ell+1}^{\ell N_s} (A+D)(\xi_\alpha). \end{aligned}\quad (7.11)$$

7.3 Useful formulas in the higher-spin case

Let us denote by $X^{\pm(\ell)}$ the matrix representations of generators X^\pm in the spin- $\ell/2$ representation of $U_q(sl_2)$. Here we recall that the matrix representations of $X^{\pm(\ell)}$ are obtained by calculating the action of $\Delta^{(\ell-1)}(X^\pm)$ on the basis $\{|\ell, n\rangle\}$.

We explicitly calculate the actions of $\sigma_1^- = \sigma^- \otimes I^{\otimes(\ell-1)}$ and $\sigma_\ell^+ = I^{\otimes(\ell-1)} \otimes \sigma^+$ on the basis $\{|\ell, n\rangle\}$ in the spin- $\ell/2$ representation. Multiplying projection operators to them, we obtain the following formulas:

$$\begin{aligned} P_{1\dots\ell}^{(\ell)} \sigma_1^- P_{1\dots\ell}^{(\ell)} &= \frac{1}{[\ell]_q} X^{-(\ell+)} \\ P_{1\dots\ell}^{(\ell)} \sigma_\ell^+ P_{1\dots\ell}^{(\ell)} &= \frac{1}{[\ell]_q} X^{-(\ell+)} \end{aligned} \quad (7.12)$$

Therefore, we have for $i = 1, 2, \dots, N_s$, the following formulas:

$$\begin{aligned} &P_{(i-1)\ell+1}^{(\ell)} X_i^{-(\ell+)} P_{(i-1)\ell+1}^{(\ell)} \\ &= [\ell]_q P_{(i-1)\ell+1}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A^+ + D^+)(\xi_\alpha) \cdot B^+(\xi_{(i-1)\ell+1}) \cdot \prod_{\alpha=(i-1)\ell+2}^{\ell N_s} (A^+ + D^+)(\xi_\alpha) P_{(i-1)\ell+1}^{(\ell)} \end{aligned} \quad (7.13)$$

$$\begin{aligned} &P_{(i-1)\ell+1}^{(\ell)} X_i^{+(\ell+)} P_{(i-1)\ell+1}^{(\ell)} \\ &= [\ell]_q P_{(i-1)\ell+1}^{(\ell)} \prod_{\alpha=1}^{i\ell-2} (A^+ + D^+)(\xi_\alpha) \cdot C^+(\xi_{i\ell-1}) \cdot \prod_{\alpha=i\ell}^{\ell N_s} (A^+ + D^+)(\xi_\alpha) P_{(i-1)\ell+1}^{(\ell)}. \end{aligned} \quad (7.14)$$

Taking advantage of projection operators, we thus have shown that the summation over k arising from the $(\ell-1)$ th comultiplication operation can be calculated by a single term. This reduces the calculational task very much.

In the derivation of (7.13), we first note

$$\chi_{1\dots L} \sigma_{(i-1)\ell+1}^- \chi_{1\dots L}^{-1} \exp(-\xi_{(i-1)\ell+1}), \quad (7.15)$$

and then we show the following:

$$\sigma_{(i-1)\ell+1}^- = \prod_{\alpha=1}^{(i-1)\ell} (A^+ + D^+)(\xi_\alpha) B^+(\xi_{(i-1)\ell+1}) \prod_{\alpha=(i-1)\ell+2}^{\ell N_s} (A^+ + D^+)(\xi_\alpha). \quad (7.16)$$

We shall show relations (7.12) in Appendix C.

We now introduce useful formulas expressing any given operator in the spin- ℓ representation. Let us take two sets of integers i_1, \dots, i_m and j_1, \dots, j_n satisfying $1 \leq i_1 < \dots < i_m \leq \ell$ and $1 \leq j_1 < \dots < j_n \leq \ell$, respectively. We can show the following:

$$\begin{aligned} ||\ell, m\rangle \langle \ell, n|| &= \left[\begin{array}{c} \ell \\ m \end{array} \right]_q q^{m(m+1)/2 - n(n-1)/2 + n\ell - (i_1 + \dots + i_m + j_1 + \dots + j_n)} \\ &\times P_{1\dots\ell}^{(\ell)} \left(\prod_{k=1}^m e_{i_k}^{21} \cdot \prod_{p=1; p \neq i_k, j_q}^{\ell} e_p^{22} \cdot \prod_{q=1}^n e_{j_q}^{12} \right) P_{1\dots\ell}^{(\ell)} \end{aligned} \quad (7.17)$$

Setting $i_1 = 1, i_2 = 2, \dots, i_m = m$ and $j_1 = 1, j_2 = 2, \dots, j_n = n$, we have for $m > n$

$$||\ell, m\rangle\langle\ell, n|| = \left[\begin{array}{c} \ell \\ n \end{array} \right]_q q^{(\ell-n)} P_{1\dots\ell}^{(\ell)} \left(\prod_{k=1}^n e_k^{22} \prod_{k=n+1}^m e_k^{21} \prod_{k=m+1}^\ell e_k^{11} \right) P_{1\dots\ell}^{(\ell)} \quad (7.18)$$

and for $m < n$

$$||\ell, m\rangle\langle\ell, n|| = \left[\begin{array}{c} \ell \\ n \end{array} \right]_q q^{(\ell-n)} P_{1\dots\ell}^{(\ell)} \left(\prod_{k=1}^m e_k^{22} \prod_{k=m+1}^n e_k^{12} \prod_{k=n+1}^\ell e_k^{11} \right) P_{1\dots\ell}^{(\ell)} \quad (7.19)$$

and for $m = n$

$$||\ell, n\rangle\langle\ell, n|| = \left[\begin{array}{c} \ell \\ n \end{array} \right]_q q^{(\ell-n)} P_{1\dots\ell}^{(\ell)} \left(\prod_{k=1}^n e_k^{22} \prod_{k=n+1}^\ell e_k^{11} \right) P_{1\dots\ell}^{(\ell)} \quad (7.20)$$

Let us denote by $E^{mn(\ell+)}$ the unit matrices acting on spin- ℓ representation $V^{(\ell)}$ for $m, n = 0, 1, \dots, \ell$. We now define $E_i^{mn(\ell+)}$ by the unit matrices acting on the i th component of the tensor product $(V^{(\ell)})^{\otimes N_s}$. Explicitly, we have

$$E_i^{mn(\ell+)} = (I^{(\ell)})^{\otimes(i-1)} \otimes E^{mn} \otimes (I^{(\ell)})^{\otimes(N_s-i)} \quad (7.21)$$

where $I^{(\ell)}$ denotes the $(\ell + 1) \times (\ell + 1)$ identity matrix. Then, we derive the following formulas from (7.18), (7.19) and (7.20), respectively. For $m > n$ we have

$$\begin{aligned} E_i^{mn(\ell+)} &= \left[\begin{array}{c} \ell \\ m \end{array} \right]_q q^{n(\ell-n)} P_{1\dots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A+D)(\xi_\alpha) \prod_{k=1}^n D(\xi_{(i-1)\ell+k}) \prod_{k=n+1}^m B(\xi_{(i-1)\ell+k}) \\ &\quad \times \prod_{k=m+1}^\ell A(\xi_{(i-1)\ell+k}) \prod_{\alpha=i\ell+1}^{\ell N_s} (A+D)(\xi_\alpha) P_{1\dots L}^{(\ell)}. \end{aligned} \quad (7.22)$$

For $m < n$ we have

$$\begin{aligned} E_i^{mn(\ell+)} &= \left[\begin{array}{c} \ell \\ m \end{array} \right]_q q^{n(\ell-n)} P_{1\dots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A+D)(\xi_\alpha) \prod_{k=1}^m D(\xi_{(i-1)\ell+k}) \prod_{k=m+1}^n C(\xi_{(i-1)\ell+k}) \\ &\quad \times \prod_{k=n+1}^\ell A(\xi_{(i-1)\ell+k}) \prod_{\alpha=i\ell+1}^{\ell N_s} (A+D)(\xi_\alpha) P_{1\dots L}^{(\ell)}. \end{aligned} \quad (7.23)$$

For $m = n$ we have

$$\begin{aligned} E_i^{nn(\ell+)} &= \left[\begin{array}{c} \ell \\ n \end{array} \right]_q q^{n(\ell-n)} P_{1\dots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A+D)(\xi_\alpha) \prod_{k=1}^n D(\xi_{(i-1)\ell+k}) \\ &\quad \times \prod_{k=n+1}^\ell A(\xi_{(i-1)\ell+k}) \prod_{\alpha=i\ell+1}^{\ell N_s} (A+D)(\xi_\alpha) P_{1\dots L}^{(\ell)}. \end{aligned} \quad (7.24)$$

Let us now discuss the derivation of formula (7.17). It is easy to show the following:

$$\sigma_{i_1}^- \cdots \sigma_{i_m}^- |0\rangle\langle 0| \sigma_{j_1}^+ \cdots \sigma_{j_n}^+ = e_{i_1}^{21} \cdots e_{i_n}^{21} \prod_{p=1; p \neq i_k, j_q}^{\ell} e_p^{11} e_{i_1}^{12} \cdots e_{i_n}^{12} \quad (7.25)$$

Then, making use of expressions (4.20) and (4.25), we obtain (7.17).

7.4 Form factors for higher-spin operators

Making use of the fundamental lemma of the quantum inverse-scattering problem, lemma 37, together with the useful formulas given in §7.3 such as (7.13) and (7.14), and (7.22), (7.23) and (7.24), we can systematically calculate form factors for the higher-spin cases. Here we note that the form factors associated with generators S^\pm of the spin $SU(2)$ have been derived for the higher-spin XXX chains [27]. They are derived through the relations corresponding to (7.8) and (7.10) in the limit of $q = 1$.

For an illustration, let us calculate the following form factor:

$$F_n^{-(\ell+)}(i; \{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^{n+1} C^{(\ell+)}(\mu_j) \cdot X_i^{-(\ell+)} \prod_{k=1}^n B^{(\ell+)}(\lambda_k) | 0 \rangle, \quad (7.26)$$

where $\{\mu_j\}$ and $\{\lambda_k\}$ are solutions of the Bethe ansatz equations. Putting (7.13) into (7.26) and making use of the fact that projector $P_{1\dots L}^{(\ell)}$ commutes with the matrix elements of $R_{0,1\dots L}^+$, we have

$$\begin{aligned} & \langle 0 | \prod_{j=1}^{n+1} C^+(\mu_j) \cdot X_i^{-(\ell+)} \prod_{k=1}^n B^+(\lambda_k) | 0 \rangle \\ &= [\ell]_q \langle 0 | \prod_{j=1}^{n+1} C^+(\mu_j) \cdot P_{(i-1)\ell+1}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A^+ + D^+)(\xi_\alpha) \cdot B^+(\xi_{(i-1)\ell+1}) \cdot \\ & \quad \times \prod_{\alpha=(i-1)\ell+2}^{\ell N_s} (A^+ + D^+)(\xi_\alpha) P_{(i-1)\ell+1}^{(\ell)} \prod_{k=1}^n B^+(\lambda_k) | 0 \rangle \\ &= [\ell]_q e^{\sum_{j=1}^{n+1} \mu_j - \sum_{k=1}^n \lambda_k - \xi_{(i-1)\ell+1}} \frac{\phi_{(i-1)\ell}(\{\mu_j\}_{n+1})}{\phi_{(i-1)\ell+1}(\{\lambda\}_n)} \langle 0 | \prod_{j=1}^{n+1} C(\mu_j) \cdot B(\xi_{(i-1)\ell+1}) \prod_{k=1}^n B(\lambda_k) | 0 \rangle \\ &= [\ell]_q e^{\sum_{j=1}^{n+1} \mu_j - \sum_{k=1}^n \lambda_k - \xi_{(i-1)\ell+1}} \frac{\phi_{(i-1)\ell}(\{\mu_j\}_{n+1})}{\phi_{(i-1)\ell+1}(\{\lambda\}_n)} \\ & \quad \times S_{n+1} \left(\{\mu_j\}_{n+1}, \{\xi_{(i-1)\ell+1}, \lambda_1, \dots, \lambda_n\}; \{\xi_k^{(\ell)}\} \right) \end{aligned} \quad (7.27)$$

Let us define the form factor in the symmetric case as follows.

$$F_n^{-(\ell)}(i; \{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^{n+1} C(\mu_j) \cdot \left(X_i^{-(\ell+)} \right)^{\bar{x}} \prod_{k=1}^n B(\lambda_k) | 0 \rangle \quad (7.28)$$

Here we recall that $\{\mu_j\}$ and $\{\lambda_k\}$ are solutions of the Bethe ansatz equations. Then, we obtain the following expression.

$$\begin{aligned} F_n^{-(\ell)}(i; \{\mu_j\}, \{\lambda_k\}) &= [\ell]_q \frac{\phi_{(i-1)\ell}(\{\mu_j\}_{n+1})}{\phi_{(i-1)\ell+1}(\{\lambda\}_n)} \\ & \quad \times S_{n+1} \left(\{\mu_j\}_{n+1}, \{\xi_{(i-1)\ell+1}, \lambda_1, \dots, \lambda_n\}; \{\xi_k^{(\ell)}\} \right). \end{aligned} \quad (7.29)$$

Let us define the form factor for K by

$$F_n^{K(\ell+)}(i, \{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^n C^{(\ell+)}(\mu_j) \cdot \left(K_i^{(\ell+)} \right)^{\bar{x}} \prod_{k=1}^n B^{(\ell+)}(\lambda_k) | 0 \rangle. \quad (7.30)$$

Here we recall that $\{\mu_j\}$ and $\{\lambda_k\}$ are solutions of the Bethe ansatz equations. Similarly, we can calculate as follows.

$$\begin{aligned} F_n^{K(\ell+)}(i, \{\mu_j\}, \{\lambda_k\}) &= \exp \left(\sum_{j=1}^n \mu_j - \sum_{k=1}^n \lambda_k \right) \frac{\phi_{(i-1)\ell}(\{\mu_j\}_n)}{\phi_{i\ell}(\{\lambda\}_n)} \\ &\times \sum_{n=0}^{\ell} \left[\begin{array}{c} \ell \\ n \end{array} \right]_q q^{\ell-2n+n(\ell-n)} \langle 0 | \prod_{j=1}^n C(\mu_j) \prod_{k=1}^n D(\xi_{(i-1)\ell+k}) \prod_{k=n+1}^{\ell} A(\xi_{(i-1)\ell+k}) \prod_{k=1}^n B(\lambda_k) | 0 \rangle \end{aligned} \quad (7.31)$$

Through the commutation relation between A and B and that between C and D , the vacuum expectation of the product of C , D , A and B operators can be expressed in terms of the sum of scalar products.

A Derivation of symmetry relations for R_p^σ

Lemma A.1. *Let p be a sequence of n integers, $1, 2, \dots, n$. For any $\sigma_A, \sigma_B \in \mathcal{S}_n$ we have*

$$(\sigma_A \sigma_B) p = \sigma_B (\sigma_A p). \quad (\text{A.1})$$

Proof. Let us denote $p_{\sigma_A i}$ by q_i for $i = 1, 2, \dots, n$. We thus have

$$\begin{aligned} \sigma_B(q_1, \dots, q_n) &= (q_{\sigma_B 1}, \dots, q_{\sigma_B n}) \\ &= (p_{\sigma_A(\sigma_B 1)}, \dots, p_{\sigma_A(\sigma_B n)}) \\ &= (p_{\sigma_A \sigma_B 1}, \dots, p_{\sigma_A \sigma_B n}) \\ &= (\sigma_A \sigma_B) p \end{aligned} \quad (\text{A.2})$$

□

Proposition A.1. *Definition 3 is well defined. That is, for $R_{j,k} = R_{j,k}(\lambda_j, \lambda_k)$ the following relations hold:*

$$R_p^{(\sigma_A \sigma_B) \sigma_C} = R_p^{\sigma_A (\sigma_B \sigma_C)} \quad (\text{A.3})$$

$$R_p^{s_j^2} = R_p^e = 1 \quad \text{for } j = 1, 2, \dots, n, \quad (\text{A.4})$$

$$R_p^{s_i s_{i+1} s_i} = R_p^{s_{i+1} s_i s_{i+1}} \quad \text{for } i = 1, 2, \dots, n-1. \quad (\text{A.5})$$

Proof. We recall that in terms of generators s_j the defining relations of the symmetric group \mathcal{S}_n are given by $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i = 1, 2, \dots, n-1$ and $s_j^2 = 1$ for $j = 1, 2, \dots, n$ [35]. It thus follows that definition 3 is well defined if and only if conditions (A.3), (A.4) and (A.5) hold. We now show them. Lemma A.1 leads to conditions (A.3). Conditions (A.4) are derived from the inversion relations (unitarity conditions) (2.8). Finally we show conditions (A.5) by the Yang-Baxter equations. □

Lemma A.2. Let p be a sequence of integers $1, 2, \dots, n$, and $R_{j,k}$ denote $R_{j,k}(\lambda_j, \lambda_k)$ for $j, k = 1, 2, \dots, n$. For $\sigma_c = (12 \cdots n)$ we have

$$R_p^{\sigma_c} = R_{p_1, p_2 \cdots p_n}. \quad (\text{A.6})$$

Proof. Noting $(12 \cdots n) = (12)(23) \cdots (n-1\ n) = s_1 s_2 \cdots s_{n-1}$. we have

$$\begin{aligned} R_p^{\sigma_c} &= R_p^{s_1 s_2 \cdots s_{n-1}} = R_{s_1 p}^{s_2 \cdots s_{n-1}} R_p^{s_1} \\ &= R_{s_2(s_1 p)}^{s_3 \cdots s_{n-1}} R_{s_1 p}^{s_2} R_p^{s_1} = R_{(s_1 s_2)p}^{s_3 \cdots s_{n-1}} R_{s_1 p}^{s_2} R_p^{s_1} \\ &= R_{(s_1 \cdots s_{n-2})p}^{s_{n-1}} \cdots R_{(s_1 s_2)p}^{s_3} R_{s_1 p}^{s_2} R_p^{s_1} \\ &= R_{(12 \cdots n-1)p}^{s_{n-1}} \cdots R_{(123)p}^{s_3} R_{(12)p}^{s_2} R_p^{s_1} \\ &= R_{p_1, p_n} \cdots R_{p_1, p_3} R_{p_1, p_2} \\ &= R_{p_1, p_2 \cdots p_n}. \end{aligned} \quad (\text{A.7})$$

□

Proposition A.2. Let \mathcal{A} be a Hopf algebra and \mathcal{R} is an element of $\mathcal{A} \otimes \mathcal{A}$ such that $\mathcal{R}\Delta(x) = \tau \circ \Delta(x)\mathcal{R}$ for all $x \in \mathcal{A}$. Suppose that \mathcal{R} is given by $\mathcal{R} = \sum_a r^{(a,1)} \otimes r^{(a,2)}$, where $r^{(a,1)}, r^{(a,2)} \in \mathcal{A}$. We define $R_{j,k}$ by

$$R_{j,k} = \sum_a id_1 \otimes \cdots \otimes r_j^{(a,1)} \otimes \cdots \otimes r_k^{(a,2)} \otimes \cdots \otimes id_n \in \mathcal{A}^{\otimes n}. \quad (\text{A.8})$$

If $R_{j,k}$ satisfy the inversion relations and the Yang-Baxter equations:

$$\begin{aligned} R_{12}R_{21} &= id, \\ R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12}, \end{aligned} \quad (\text{A.9})$$

then we have the following symmetry relations:

$$R_{p_q}^\sigma \Delta^{(n-1)}(x) = \sigma \circ \Delta^{(n-1)}(x) R_{p_q}^\sigma \quad x \in \mathcal{A}. \quad (\text{A.10})$$

Here p_q denotes $p_q = (1, 2, \dots, n)$, a sequence of n integers, and we have defined $R_{p_q}^\sigma$ as in definition 3.

Proof. Recall that any given permutation σ is expressed as a product of generators $s_j = (j\ j+1)$. We show symmetry relations (A.10) by induction on the number of generators s_j whose product gives permutation σ . Suppose that (A.10) holds for $\sigma = \sigma_A$. We now show that (A.10) holds for $\sigma = \sigma_A s_j$. Making use of eq. (2.21) in definition 3 we have

$$R_{p_q}^{\sigma_A s_j} \Delta^{(n-1)}(x) = R_{\sigma_A(p_q)}^{s_j} R_{p_q}^{\sigma_A} \Delta^{(n-1)}(x) = R_{\sigma_A(p_q)}^{s_j} \sigma_A \circ \Delta^{(n-1)}(x) R_{p_q}^{\sigma_A} \quad (\text{A.11})$$

Expressing $\Delta^{(n-1)}(x)$ as $\Delta^{(n-1)}(x) = \sum x^{(1)} \otimes x^{(2)} \otimes \cdots \otimes x^{(n)}$, we have

$$\sigma_A \circ \Delta^{(n-1)}(x) = \sum x^{(\sigma_A^{-1}1)} \otimes x^{(\sigma_A^{-1}2)} \otimes \cdots \otimes x^{(\sigma_A^{-1}n)} \quad (\text{A.12})$$

It now follows from definition 3 that we have $R_{\sigma_A(p_q)}^{s_j} = R_{\sigma_A(j), \sigma_A(j+1)}$. Let us denote $\sigma_A j$ and $\sigma_A(j+1)$ by a and b , respectively. Then we have $\sigma_A^{-1}a = j$ and $\sigma_A^{-1}b = j+1$. Assuming $a < b$ we have

$$\begin{aligned} & R_{\sigma_A(j), \sigma_A(j+1)} \sigma_A \circ \Delta^{(n-1)}(x) \\ = & R_{a,b} \sum x_1^{(\sigma_A^{-1}1)} \otimes x_2^{(\sigma_A^{-1}2)} \otimes \cdots \otimes x_a^{(\sigma_A^{-1}a)} \otimes \cdots \otimes x_b^{(\sigma_A^{-1}b)} \otimes \cdots \otimes x_n^{(\sigma_A^{-1}n)} \\ = & \sum x_1^{(\sigma_A^{-1}1)} \otimes x_2^{(\sigma_A^{-1}2)} \otimes \cdots \otimes x_a^{(\sigma_A^{-1}b)} \otimes \cdots \otimes x_b^{(\sigma_A^{-1}a)} \otimes \cdots \otimes x_n^{(\sigma_A^{-1}n)} R_{a,b} \\ = & (\sigma_A s_j) \circ \Delta^{(n-1)}(x) R_{\sigma_A(j), \sigma_A(j+1)} \end{aligned} \quad (\text{A.13})$$

Here we note that

$$R_{a,b} x_a^{(\sigma_A^{-1}a)} \otimes x_b^{(\sigma_A^{-1}b)} = x_a^{(\sigma_A^{-1}b)} \otimes x_b^{(\sigma_A^{-1}a)} R_{a,b} \quad (\text{A.14})$$

since we have

$$x^{(\sigma_A^{-1}a)} \otimes x^{(\sigma_A^{-1}b)} = x^{(j)} \otimes x^{(j+1)} = \Delta(\bar{x}^{(j)})$$

where $\bar{x}^{(j)}$ are defined by

$$\Delta^{(n-2)}(x) = \sum \bar{x}^{(1)} \otimes \cdots \otimes \bar{x}^{(n-1)}$$

We therefore have

$$\begin{aligned} R_{\sigma_A(p_q)}^{s_j} \sigma_A \circ \Delta^{(n-1)}(x) R_{p_q}^{\sigma_A} &= (\sigma_A s_j) \circ \Delta^{(n-1)}(x) R_{\sigma_A(p_q)}^{s_j} R_{p_q}^{\sigma_A} \\ &= (\sigma_A s_j) \circ \Delta^{(n-1)}(x) R_{p_q}^{\sigma_A s_j} \end{aligned} \quad (\text{A.15})$$

□

Symmetry relations similar to (A.10) hold for products of monodromy matrices. Let us consider m auxiliary spaces with suffices $a(1), a(2), \dots, a(m)$, respectively. We denote the monodromy matrix $T_{a(j)}(\lambda_{a(j)}; \xi_1, \dots, \xi_L)$ simply by $T_{a(j)}$. We denote by $\Delta^{(m-1)}(T)$ the following operator:

$$\Delta^{(m-1)}(T) = T_{a(1)} T_{a(2)} \cdots T_{a(m)} \quad (\text{A.16})$$

Let σ an element of \mathcal{S}_m . We define the action of σ on $\Delta^{(m-1)}(T)$ by the following:

$$\sigma \circ \Delta^{(m-1)}(T) = T_{a(\bar{\sigma}1)} T_{a(\bar{\sigma}2)} \cdots T_{a(\bar{\sigma}m)} \quad (\text{A.17})$$

Here $\bar{\sigma}$ denotes the inverse of σ : $\bar{\sigma} = \sigma^{-1}$. Then we have the following.

Proposition A.3. *Let p_q be $p_q = (1, 2, \dots, m)$. For any $\sigma \in \mathcal{S}_m$ we have*

$$R_{p_q}^{\sigma} \Delta^{(m-1)}(T) = \sigma \circ \Delta^{(m-1)}(T) R_{p_q}^{\sigma} \quad (\text{A.18})$$

B Symmetric-group action on products of R -matrices and the F -basis

Lemma B.1. ([6]) (i) Cocycle conditions hold for $n \leq L$.

$$R_{2\cdots n-1, n} R_{1, 2\cdots n} = R_{1, 2\cdots n-1} R_{12\cdots n-1, n} \quad (\text{B.1})$$

(ii) The unitarity relations hold for $n \leq L$.

$$R_{1, 2\cdots n} R_{23\cdots n, 1} = I^{\otimes L} \quad (\text{B.2})$$

Proof. Cocycle conditions (B.1) are derived from the Yang-Baxter equations. \square

Let us denote by the symbol $(p_0, p_1, p_2, \dots, p_n)$ a sequence of $n+1$ integers, $0, 1, 2, \dots, n$, and we express it as (p_0, p) where p denotes the subsequence (p_1, p_2, \dots, p_n) .

Lemma B.2. Let p be a sequence of n integers, $1, 2, \dots, n$. For $s_j = (j \ j+1) \in \mathcal{S}_n$ we have

$$\begin{aligned} R_p^{s_j} R_{0, p} &= R_{0, s_j(p)} R_p^{s_j} \\ R_p^{s_j} R_{p, 0} &= R_{s_j(p), 0} R_p^{s_j} \end{aligned} \quad (\text{B.3})$$

Proof. We first note $R_p^{s_j} = R_{p_j, p_{j+1}}$. We have

$$\begin{aligned} R_p^{s_j} R_{0, p} &= R_{p_j, p_{j+1}} R_{0, p_n} \cdots R_{0, p_1} \\ &= R_{0, p_n} \cdots R_{0, p_{j+2}} \cdot R_{p_j, p_{j+1}} R_{0, p_{j+1}} R_{0, p_j} \cdot R_{0, p_{j+2}} \cdots R_{0, p_1} \end{aligned} \quad (\text{B.4})$$

Applying the Yang-Baxter equations $R_{p_j, p_{j+1}} R_{0, p_{j+1}} R_{0, p_j} = R_{0, p_j} R_{0, p_{j+1}} R_{p_j, p_{j+1}}$ we have

$$\begin{aligned} R_p^{s_j} R_{0, p} &= R_{0, p_n} \cdots R_{0, p_{j+2}} \cdot R_{0, p_j} R_{0, p_{j+1}} \cdot R_{0, p_{j-1}} \cdots R_{0, p_1} \cdot R_{p_j, p_{j+1}} \\ &= R_{0, s_j(p)} R_{p_j, p_{j+1}}. \end{aligned} \quad (\text{B.5})$$

\square

Proposition B.1. Let p be a sequence of n integers, $1, 2, \dots, n$, and σ a permutation of the n integers. We have

$$\begin{aligned} R_p^\sigma R_{0, p} &= R_{0, \sigma(p)} R_p^\sigma \\ R_p^\sigma R_{p, 0} &= R_{\sigma(p), 0} R_p^\sigma \end{aligned} \quad (\text{B.6})$$

Proof. Expressing permutation σ as a product of generators s_j , and applying (B.3) many times, we can show the symmetry relations. \square

We define the action of σ on the F -basis as follows.

$$F_{\sigma(p)} = F_{p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(n)}} \quad (\text{B.7})$$

Proposition B.2. Let p be a sequence of integers, $1, 2, \dots, n$. For $\sigma \in \mathcal{S}_n$ we have

$$R_p^\sigma F_{0,p} = F_{0,\sigma(p)} R_p^\sigma \quad (\text{B.8})$$

We also have

$$F_p = F_{\sigma(p)} R_p^\sigma \quad (\text{B.9})$$

Proof. Expressing the F -basis in terms of the R -matrices we have

$$\begin{aligned} R_p^\sigma F_{0,p} &= R_p^\sigma (e_0^{11} + e_0^{22} R_{0,p}^\sigma) \\ &= e_0^{11} R_p^\sigma + e_0^{22} R_p^\sigma R_{0,p}^\sigma \\ &= e_0^{11} R_p^\sigma + e_0^{22} R_{0,\sigma(p)}^\sigma R_p^\sigma \\ &= (e_0^{11} + e_0^{22} R_{0,\sigma(p)}^\sigma) R_p^\sigma \\ &= F_{0,\sigma(p)} R_p^\sigma \end{aligned} \quad (\text{B.10})$$

We first show (B.9) with $\sigma = s_j$ for $j = 1, 2, \dots, n-1$, and then we derive it for all permutations σ . \square

Lemma B.3. The propagator $F_{i \dots L 1 \dots i-1}^{-1} F_{1 \dots L}$ is given by the following:

$$F_{i \dots L 1 \dots i-1}^{-1} F_{1 \dots L} = \prod_{\alpha=1}^{i-1} (A_{1 \dots L}(\xi_\alpha) + D_{1 \dots L}(\xi_\alpha)) \quad (\text{B.11})$$

Proof. Let σ_c be a cyclic permutation: $\sigma_c = (12 \dots L)$, and p_q the sequence $p_q = (1, 2, \dots, n)$. We have

$$F_{i \dots L 1 \dots i-1} = F_{\sigma_c^{i-1}(p_q)} = F_{1 \dots L} R_{p_q}^{\sigma_c^{i-1}} \quad (\text{B.12})$$

and hence we have

$$\begin{aligned} F_{i \dots L 1 \dots i-1}^{-1} F_{1 \dots L} &= \left(F_{1 \dots L} R_{p_q}^{\sigma_c^{i-1}} \right)^{-1} F_{1 \dots L} \\ &= R_{p_q}^{\sigma_c^{i-1}} F_{1 \dots L}^{-1} F_{1 \dots L} \\ &= R_{p_q}^{\sigma_c^{i-1}} \end{aligned} \quad (\text{B.13})$$

We thus obtain

$$\begin{aligned} R_{p_q}^{\sigma_c^{i-1}} &= R_{\sigma_c(p_q)}^{\sigma_c^{i-2}} R_{p_q}^{\sigma_c} \\ &= R_{\sigma_c^{i-2}(p_q)}^{\sigma_c} \cdots R_{p_q}^{\sigma_c} \\ &= R_{i-1 \dots L 1 \dots i-2}^{\sigma_c} \cdots R_{2 \dots L 1}^{\sigma_c} R_{12 \dots L}^{\sigma_c} \\ &= R_{i-1, i \dots L 1 \dots i-2} \cdots R_{2, 3 \dots L 1} R_{1, 2 \dots L} \\ &= \prod_{\alpha=1}^{i-1} (A_{1 \dots L}(\xi_\alpha) + D_{1 \dots L}(\xi_\alpha)) \end{aligned}$$

\square

C Formulas of the q -analogue

Lemma C.1. *For two integers ℓ and n satisfying $0 \leq n \leq \ell$ we have*

$$\sum_{1 \leq i_1 < \dots < i_n \leq \ell} q^{2i_1 + \dots + 2i_n} = q^{n(\ell+1)} \begin{bmatrix} \ell \\ n \end{bmatrix}_q \quad (\text{C.1})$$

Proof. We can show by induction on ℓ the q -binomial expansion as follows.

$$\prod_{k=0}^{\ell-1} (1 - zq^{2k}) = \sum_{n=0}^{\ell} (-z)^n q^{n(\ell-1)} \begin{bmatrix} \ell \\ n \end{bmatrix}_q \quad (\text{C.2})$$

It is now easy to show the following:

$$\prod_{k=1}^{\ell} (1 - zq^{2k}) = \sum_{n=0}^{\ell} (-z)^n \sum_{1 \leq i_1 < \dots < i_n \leq \ell} q^{2i_1 + \dots + 2i_n} \quad (\text{C.3})$$

Comparing (C.3) with (C.2) we obtain formula (C.1). \square

Lemma C.2. *The spin- ℓ matrix representations $X^{\pm(\ell+)}$ of the generators X^\pm of $U_q(sl_2)$ are related to σ_ℓ^+ and σ_1^- , respectively, as follows.*

$$P_{1\dots\ell}^{(\ell)} \sigma_1^- P_{1\dots\ell}^{(\ell)} = \frac{1}{[\ell]_q} X^{-(\ell+)} \quad (\text{C.4})$$

$$P_{1\dots\ell}^{(\ell)} \sigma_\ell^+ P_{1\dots\ell}^{(\ell)} = \frac{1}{[\ell]_q} X^{+(\ell+)} \quad (\text{C.5})$$

Proof. Expressing the projection operator $P_{1\dots\ell}^{(\ell)}$ as

$$P_{1\dots\ell}^{(\ell)} = \sum_{n=0}^{\ell} |\ell, n\rangle \langle \ell, n|$$

we have

$$P_{1\dots\ell}^{(\ell)} \sigma_1^- P_{1\dots\ell}^{(\ell)} = \sum_{n=0}^{\ell-1} |\ell, n+1\rangle \langle \ell, n| - \langle \ell, n+1 | \sigma_1^- | \ell, n \rangle \quad (\text{C.6})$$

Making use of (C.1) we have

$$\langle \ell, n+1 | \sigma_1^- | \ell, n \rangle = \frac{[n+1]}{[\ell]}, \quad (\text{C.7})$$

and then we obtain (C.4). Similarly, we can show (C.5). \square

D Formulas of the F -basis useful for the diagonalization.

Let us review some points of the diagonalization process of the A and D operators [6].

Lemma D.1. Operators A and D are upper- and lower-triangular matrices, respectively. Moreover, the eigenvalues of operators A and D are given by

$$\text{diag}(D_{12\cdots n}(\lambda_0)) = \bigotimes_{i=1}^n \begin{pmatrix} b_{0i} & 0 \\ 0 & 1 \end{pmatrix}_{[i]}, \quad (\text{D.1})$$

$$\text{diag}(A_{12\cdots n}(\lambda_0)) = \bigotimes_{i=1}^n \begin{pmatrix} 1 & 0 \\ 0 & b_{0i} \end{pmatrix}_{[i]}, \quad (\text{D.2})$$

where $b_{0i} = b(\lambda_0 - \xi_i)$.

Proof. We can show it by induction on n . Noting $D_{12\cdots n} = C_n B_{1\cdots n-1} + D_n D_{1\cdots n-1}$, we show $\text{diag}(D_{1\cdots n}) = \text{diag}(D_{1\cdots n-1}(\lambda_0) \otimes \text{diag}(b_{0n}, 1)_{[n]})$. We can show (D.2) similarly. \square

Lemma D.2. The partial $F_{0,1\cdots n}$ and $\bar{F}_{0,1\cdots n}$ are expressed as follows.

$$F_{0,1\cdots n} = \begin{pmatrix} I_{1\cdots n} & 0 \\ C_{1\cdots n}(\lambda_0) & D_{1\cdots n}(\lambda_0) \end{pmatrix}_{[0]}, \quad (\text{D.3})$$

$$\bar{F}_{0,1\cdots n} = \begin{pmatrix} A_{1\cdots n}(\lambda_0) & B_{1\cdots n}(\lambda_0) \\ 0 & I_{1\cdots n} \end{pmatrix}_{[0]}. \quad (\text{D.4})$$

Proof. It is clear from definition 15 of the F -basis. \square

Proof of propositions 25 and 26

For an illustration, we now derive the diagonalized form of the D operator. From $R_{0,1\cdots n} = R_{0,2\cdots n}R_{0,1}$ we have

$$\begin{aligned} D_{12\cdots n} &= C_{2\cdots n}B_1 + D_{2\cdots n}D_1 \\ &= \begin{pmatrix} b_{01}D_{2\cdots n}(\lambda_0) & 0 \\ c_{01}C_{2\cdots n}(\lambda_0) & D_{2\cdots n}(\lambda_0) \end{pmatrix}_{[0]}. \end{aligned} \quad (\text{D.5})$$

We thus calculate

$$\begin{aligned} F_{1\cdots n}D_{1\cdots n}(\lambda_0) &= F_{2\cdots n}F_{1,2\cdots n}D_{1\cdots n}(\lambda_0) \\ &= F_{2\cdots n}\begin{pmatrix} I_{2\cdots n} & 0 \\ C_{2\cdots n}(\lambda_0) & D_{2\cdots n}(\lambda_0) \end{pmatrix}_{[0]}\begin{pmatrix} b_{01}D_{2\cdots n}(\lambda_0) & 0 \\ c_{01}C_{2\cdots n}(\lambda_0) & D_{2\cdots n}(\lambda_0) \end{pmatrix}_{[0]} \\ &= \text{diag}(b_{01}, 1)_{[1]}\tilde{D}_{2\cdots n}(\lambda_0)F_{1\cdots n} \end{aligned} \quad (\text{D.6})$$

Therefore, by induction we have the diagonalized form of operator D . Similarly, we can diagonalize A , A^\dagger and D^\dagger .

Lemma D.3. The diagonalized form of $F_{0,1\cdots n}\bar{F}_{0,1\cdots n}^\dagger$ is given by the following:

$$F_{1\cdots n} \left(F_{0,1\cdots n}\bar{F}_{0,1\cdots n}^\dagger \right) F_{1\cdots n}^{-1} = \hat{\delta}_{0,1\cdots n}^{-1} \quad (\text{D.7})$$

Proof. Making use of (5.12) we show

$$\begin{aligned}
F_{0,1 \dots n} \bar{F}_{0,1 \dots n}^\dagger &= F_{0,1 \dots n} \mathcal{C}_{01 \dots n} F_{0,1 \dots n}^\dagger \mathcal{C}_{01 \dots n} \\
&= \begin{pmatrix} 1 & 0 \\ C_{1 \dots n} & D_{1 \dots n} \end{pmatrix}_{[0]} \begin{pmatrix} 0 & \mathcal{C}_{1 \dots n} \\ \mathcal{C}_{1 \dots n} & 0 \end{pmatrix}_{[0]} \begin{pmatrix} 1 & C_{1 \dots n}^\dagger \\ 0 & D_{1 \dots n} \end{pmatrix}_{[0]} \begin{pmatrix} 0 & \mathcal{C}_{1 \dots n} \\ \mathcal{C}_{1 \dots n} & 0 \end{pmatrix}_{[0]} \\
&= \begin{pmatrix} A^\dagger & 0 \\ 0 & D_{1 \dots n} \end{pmatrix}_{[0]}.
\end{aligned}$$

Calculating $F_{12 \dots n} \cdot F_{0,1 \dots n} \bar{F}_{0,1 \dots n}^\dagger \cdot F_{12 \dots n}^{-1}$ we have (D.7). \square

Corollary D.1. *The inverse of the total F is given as follows.*

$$F_{1 \dots n}^{-1} = \bar{F}_{1 \dots n}^\dagger \hat{\delta}_{1 \dots n} \quad (\text{D.8})$$

Proof. We show it by induction on n . Let us assume (D.8) for the case of n . We have

$$\bar{F}_{01 \dots n}^\dagger \hat{\delta}_{01 \dots n} = \bar{F}_{01 \dots n}^\dagger \left(\bar{F}_{1 \dots n}^\dagger \hat{\delta}_{1 \dots n} \right) \hat{\delta}_{0,1 \dots n} = \bar{F}_{01 \dots n}^\dagger F_{1 \dots n}^{-1} \hat{\delta}_{0,1 \dots n}$$

From (D.7) we have $\bar{F}_{01 \dots n}^\dagger F_{1 \dots n}^{-1} \hat{\delta}_{0,1 \dots n} = F_{01 \dots n}^{-1}$, which corresponds to (D.8) for the case of $n + 1$. \square

Lemma D.4. *The dagger of total F is given by the charge conjugation of transposed total F .*

$$F_{1 \dots n}^\dagger = \mathcal{C}_{1 \dots n} F_{n \dots 21}^{t_1 \dots t_n} \mathcal{C}_{1 \dots n} \quad (\text{D.9})$$

Or equivalently we have

$$\bar{F}_{1 \dots n}^\dagger = F_{1 \dots n}^{t_1 \dots t_n} \quad (\text{D.10})$$

Proof. We show it by induction on n . Let us assume (D.9) for the case of $n - 1$. We first show

$$\bar{F}_{1,2 \dots n}^\dagger = \mathcal{C}_{12 \dots n} (e_1^{11} + e_1^{22} R_{1,2 \dots n})^\dagger \mathcal{C}_{12 \dots n} = F_{n \dots 2,1}^{t_1 \dots t_n}.$$

Making use of the induction assumption we show

$$\begin{aligned}
F_{12 \dots n}^\dagger &= (F_{2 \dots n} F_{1,2 \dots n})^\dagger = F_{1,2 \dots n}^\dagger F_{2 \dots n}^\dagger \\
&= \mathcal{C}_{1 \dots n} F_{n \dots 2,1}^{t_1 \dots t_n} \mathcal{C}_{1 \dots n} F_{n \dots 2}^{t_2 \dots t_n} \mathcal{C}_{1 \dots n} \\
&= \mathcal{C}_{1 \dots n} F_{n \dots 21}^{t_1 \dots t_n} \mathcal{C}_{1 \dots n}.
\end{aligned}$$

\square

Corollary D.2. *The inverse of total F is given as follows.*

$$F_{1 \dots n} F_{n \dots 21}^{t_1 \dots t_n} = \hat{\delta}_{1 \dots n}^{-1} \quad (\text{D.11})$$

Equivalently, we have

$$F_{1 \dots n}^{-1} = F_{n \dots 21}^{t_1 \dots t_n} \hat{\delta}_{1 \dots n} \quad (\text{D.12})$$

Proof. It follows from (D.8) that $\bar{F}^\dagger = F_{n \dots 1}^{t_1 \dots t_n} \hat{\delta}_{12 \dots n}$. From (D.10) we have (D.11). \square

E Lemmas for Diagonalizing the C operator

Lemma E.1. Let X^+ be the generator of the quantum group $U_q(sl_2)$ and X_i^+ the matrix representation of X^+ acting on the i th site. We have

$$\widetilde{\Delta}_{1\dots n}(X^+) = \left(X_n^+ + e_n^{11} \widetilde{A}^+_{1\dots n-1}(\xi_n) \widetilde{\Delta}_{1\dots n-1}(X^+) + e_n^{22} \widetilde{\Delta}_{1\dots n-1}(X^+) \widetilde{D}^+_{1\dots n-1}(\xi_n) \right) \widehat{\delta}_n^{12\dots n}. \quad (\text{E.1})$$

Lemma E.2. The diagonalized form of $\Delta^{(n-1)}(X^+)$ is expressed in terms of local operators X_i^+ as follows.

$$\widetilde{\Delta}_{1\dots n}(X^+) = \sum_{i=1}^n X_i^+ \widehat{\delta}_i^{1\dots n} \quad (\text{E.2})$$

Lemma E.3. Operator C^+ in the F -basis is expressed in terms of operator D^+ and X_i^+ as follows.

$$\widetilde{C}^+_{1\dots n}(\lambda) = \sum_{i=1}^n (\widetilde{D}^+_{1\dots i-1, i+1\dots n}(\lambda) - q^{-1} \widetilde{D}^+_{1\dots n}(\lambda)) X_i^+ \widehat{\delta}_i^{1\dots n}. \quad (\text{E.3})$$

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